Open

Number Theory

Test #641

Directions:

1. Fill out the top left section of the scantron. Do not abbreviate your school name.

2. In the Student ID Number grid, write your 9-digit ID# and bubble.

3. In the Test Code grid, write the 3-digit test# on this test cover and bubble.

4. Scoring for this test is 5 times the number correct plus the number omitted.

5. TURN OFF ALL CELL PHONES.

6. No calculators may be used on this test.

7. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

8. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

9. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

10. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

11. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

12. Unless a question asks for an approximation or a rounded answer, give the exact answer.

For all questions, the answer choice "E. NOTA" means none of the above answers is correct. $\mathbb{N}$ *refers to the set of positive integers* $\{1,2,3,\ldots\}$. *The notation,* $\gcd(a,b)$, *refers to the greatest common divisor of* a *and* b. *Good luck and have fun!*

1. How many nonnegative integers less than 420 are divisible by 2 or 3?
   A. 140 B. 210 C. 280 D. 350 E. NOTA

2. Which of the following numbers is the smallest: $2^{2020}, 3^{1212}, 4^{1616}, 5^{808}$?
   A. 2^{2020} B. 3^{1212} C. 4^{1616} D. 5^{808} E. NOTA

3. How many positive divisors does 2020 have?
   A. 4 B. 6 C. 8 D. 10 E. NOTA

4. What is the sum of factors of 21600 which are perfect squares?
   A. 5180 B. 5320 C. 5460 D. 5600 E. NOTA

5. For how many integers $1 \leq n \leq 30$ does n have exactly 4 positive divisors?
   A. 7 B. 8 C. 9 D. 10 E. NOTA

6. What is the product of the positive divisors of 225?
   A. 15^6 B. 15^7 C. 15^8 D. 15^9 E. NOTA

For questions 7-11:
Any real number x can be represented using any **base** $b \in \mathbb{Z}$ and $|b| > 1$ by using the digits $0,1,\ldots,|b|-1$. The base b representation of x may be written as

$$(a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots)_b = \sum_{r=-\infty}^{n} a_r \cdot b^r = x$$

7. For what positive value of n is $132_n = 110_7$?
A. 3 B. 4 C. 5 D. 6 E. NOTA

8. What is the sum of digits of the base -2 representation of -11_{10}?
A. 3 B. 4 C. 5 D. 6 E. NOTA

9. Which of the following integers in base 7 is divisible by 8?
A. 1314335_7 B. 3650235_7 C. 10582414_7 D. 11232464_7 E. NOTA

10. How many trailing zeros are at the end of the base 12 expansion of $100!_{10}$?
A. 44 B. 45 C. 46 D. 47 E. NOTA

11. Let f be a polynomial with nonnegative integer coefficients. If $f(1) = 6$ and $f(6) = 1766$, what is $f(10)$?
A. 12102 B. 12120 C. 12201 D. 21012 E. NOTA

12. Let $m, n \in \mathbb{N}$ such that $7m + 5n = 200$. What is the minimum possible value of $m + n$?
A. 32 B. 34 C. 36 D. 38 E. NOTA

13. Let $x, y \in \mathbb{N}$ such that $12x^3 = y^5$. What is the minimum possible value of $x + y$?

A. 64 B. 68 C. 72 D. 76 E. NOTA

14. What is the units digit of 7^{2020}?

A. 1 B. 3 C. 7 D. 9 E. NOTA

15. What is the remainder when 4^{2020} is divided by 17?

A. 1 B. 4 C. 13 D. 16 E. NOTA

16. How many integer pairs (a, b) are there such that $\frac{7}{a} + \frac{3}{b} = 1$?

A. 1 B. 2 C. 4 D. 8 E. NOTA

17. Let $x, y, z \in \mathbb{Z}$ such that $x^2 + y^2 = z^2$. What is the largest integer value of d such that $d|xyz$ for all such x, y, z?

A. 12 B. 24 C. 30 D. 60 E. NOTA

18. Suppose that $3 \cdot x$ leaves a remainder of 7 when divided by 16. What is the remainder when x is divided by 16?

A. 12 B. 13 C. 14 D. 15 E. NOTA

19. How many ordered triples of nonnegative integers (x, y, z) satisfy the equation

$$2x + y + z = 8$$

A. 25 B. 26 C. 27 D. 28 E. NOTA

20. What is the largest prime factor of $2^{18} - 1$?

A. 3 B. 7 C. 41 D. 73 E. NOTA

For questions 21-24:
Euler's Totient Function denoted by $\phi(n)$ is equal to the number of positive integers less than or equal to n which are relatively prime to n.

21. What is the value of $\phi(84)$?

A. 18 B. 21 C. 24 D. 30 E. NOTA

22. What is the sum of all n such that $\phi(n) = 6$?

A. 27 B. 34 C. 39 D. 48 E. NOTA

23. What is the number of ordered positive integer pairs (m, n) with $m + n = 500$ such that $\gcd(m, n) = 1$?

A. 100 B. 150 C. 200 D. 250 E. NOTA

24. Let $1 \leq n \leq 2500$ be an integer which minimizes the value of $\frac{\phi(n)}{n}$. What is the sum of digits of n?

A. 5 B. 6 C. 7 D. 8 E. NOTA

25. What is $\gcd(73824,6432)$?

A. 48 B. 72 C. 96 D. 120 E. NOTA

26. For how many positive integer values of n is $\frac{6n+5}{9n+8}$ reducible?

A. 1 B. 2 C. 3 D. 4 E. NOTA

27. What is the sum of all positive integers d such that $d|(3n^2+1)$ and $d|(n^2+2n+4)$ for at least one $n \in \mathbb{Z}$?

A. 40 B. 80 C. 120 D. 160 E. NOTA

28. Let $x \in \mathbb{N}$ such that $x(x+2)(x+4) = 438672$. What is the sum of digits of x?

A. 11 B. 12 C. 13 D. 14 E. NOTA

29. If $p, q,$ and r are distinct prime numbers such that $pqr = (p+q+5)(q+r+4)$, then what is $pq+qr+pr$?

A. 309 B. 311 C. 313 D. 315 E. NOTA

30. What is the sum of all positive integers n such that $(2^n+n)|(8^n+n)$?

A. 5 B. 7 C. 11 D. 13 E. NOTA