Alpha Bowl Answers and Solutions

ANSWERS

0. 135
1. 5π
2. 3
3. 15
4. 3
5. 2
6. 126
7. 61
8. 73
9. 17
10. 25
11. ($)39,100
12. $-\frac{\sqrt{2}}{2}$
13. 8
14. 51(%)

SOLUTIONS

Question 0

$x=\lfloor x\rfloor+\{x\}$ and $y=\lfloor y\rfloor+\{y\}\to\begin{cases}\lfloor x\rfloor+\lfloor y\rfloor+\lfloor y\rfloor+\{y\}=19.8\\ \lfloor x\rfloor+\{x\}+\lfloor y\rfloor+\{y\}-\lfloor x\rfloor=8.5\end{cases}\to\begin{cases}\lfloor x\rfloor+2\lfloor y\rfloor+\{y\}=19.8\\ \lfloor y\rfloor+\{x\}+\{y\}=8.5\end{cases}\to$

$\{y\}=0.8\to\lfloor y\rfloor+\{x\}+\{y\}=8.5\to\lfloor y\rfloor+\{x\}=7.7\to\{x\}=0.7\to\begin{cases}\lfloor x\rfloor+2\lfloor y\rfloor=19\\ \lfloor y\rfloor=7\end{cases}\to\lfloor x\rfloor=5\to$

$x=5.7,\ y=7.8\to10(13.5)=135.$

Question 1

A = 2, B = 3π: $\csc x+\sin 2x=\tan\frac{x}{2}+2\to\frac{2\sin x\cos x}{\sin x}=\frac{(1-\cos x)+2\sin x}{\sin x}\to$

$2\sin x\cos x=1-\cos x+2\sin x\to2\sin x(\cos x-1)+1(\cos x-1)=0\to$

$(2\sin x+1)(\cos x-1)=0\to x=\frac{7}{6}\pi,\ \frac{11}{6}\pi,\ 0$ but $\csc 0$ is undefined. $\frac{11}{6}\pi+\frac{7}{6}\pi=3\pi$

C = $\frac{\pi}{2}$: In a right triangle, the cosine of one acute angle is the sine of the other acute angle, so these angles add to $\frac{\pi}{2}$.

D = $\begin{cases}2\sin^2\theta-\cos2\theta=0\\ 2\cos^2\theta-3\sin\theta=0\end{cases}\to\begin{cases}2\sin^2\theta-\left(1-2\sin^2\theta\right)=0\\ 2\left(1-\sin^2\theta\right)-3\sin\theta=0\end{cases}\to\begin{cases}4\sin^2\theta-1=0\\ 2\sin^2\theta+3\sin\theta-2=0\end{cases}\to$

$\begin{cases}\sin\theta=\pm\frac{1}{2}\\ \left(2\sin\theta-1\right)\left(\sin\theta+2\right)=0\end{cases}\to\sin\theta=\frac{1}{2}\to\theta=\frac{\pi}{6},\ \frac{5\pi}{6}\to\frac{\pi}{6}+\frac{5\pi}{6}=\pi$

$2\left(\frac{\pi}{2}\right)+3\pi+\pi=5\pi$

Question 2

A = $\frac{4}{3}$: $\frac{x^2}{4}-\frac{y^2}{b^2}=1\to\frac{16}{4}-\frac{4}{b^2}=1\to b^2=\frac{4}{3}.\ a^2+b^2=c^2\to4+\frac{4}{3}=c^2=\frac{16}{3}\to e^2=\frac{c^2}{a^2}=\frac{\left(\frac{16}{3}\right)}{5}=\frac{4}{3}$

B = 4: $\begin{cases}x^2+y^2-2x-2y-2=0\\ x^2+y^2-6x-6y+14=0\end{cases}\to4x+4y-16=0\to x+y=4\to y=4-x.$

$x^2+(4-x)^2-2x-2(4-x)-2=0\to x^2-4x+3=0\to(x-3)(x-1)=0\to(1,\ 3),\ (3,\ 1);$
centers are (1, 1), (3, 3). Quadrilateral is a square of area 4.

$\frac{B}{A}=\frac{4}{\left(\frac{4}{3}\right)}=3$

Question 3

A = 0: $z=1-i\rightarrow|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}\rightarrow\left(\sqrt{2}\right)^x=2^x\rightarrow 2^{\frac{1}{2}x}=2^x\rightarrow x=0,$ so there are no non-zero integral solutions.

B = 10π: $\frac{3+2i\sin\theta}{1-2i\sin\theta}\bullet\frac{1+2i\sin\theta}{1+2i\sin\theta}=\frac{3-4\sin^2\theta}{1+4\sin^2\theta}+\left(\frac{8\sin\theta}{1+4\sin^2\theta}\right)i.$ We need the imaginary part to equal 0, so set the numerator equal to 0 and solve: $8\sin\theta=0\rightarrow\theta=0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ldots$ Sum of four smallest positive values is $10\pi.$ (None of these cause the denominator to also be 0.)

C = $\frac{2}{3}\pi$: Using the work for part B, we need the real part to equal 0: $\frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0\rightarrow\sin^2\theta=\frac{3}{4}\rightarrow$ $\sin\theta=\pm\frac{\sqrt{3}}{2}\rightarrow\theta=-\frac{\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}\rightarrow\text{sum is }\frac{2\pi}{3}$

D = 1: $x+iy=\frac{a+bi}{a-bi}\bullet\frac{a+bi}{a+bi}=\frac{a^2-b^2}{a^2+b^2}+\left(\frac{2ab}{a^2+b^2}\right)i.$ $\left(\frac{a^2-b^2}{a^2+b^2}\right)^2+\left(\frac{2ab}{a^2+b^2}\right)^2=\frac{a^4-2a^2b^2+b^4+4a^2b^2}{\left(a^2+b^2\right)^2}=$ $\frac{\left(a^2+b^2\right)^2}{\left(a^2+b^2\right)^2}=1$

$\frac{A+B}{CD}=\frac{0+10\pi}{\left(\frac{2}{3}\pi\right)(1)}=15$

Question 4

A = 14: Using ratios between equations, $k=14.$

B = 9: Using ratios between x- and y-coefficients, $k=9.$

C = 2: Use row reduction: $\begin{bmatrix}-3&2&-4&6\\7&6&4&k\\-5&-4&-3&-1\end{bmatrix}\rightarrow(-2R1+R3\rightarrow R3)\rightarrow\begin{bmatrix}-3&2&-4&6\\13&2&12&k-12\\1&-8&5&-13\end{bmatrix}\rightarrow$

$$\begin{pmatrix}R1\rightarrow R2\\R2\rightarrow R3\\R3\rightarrow R1\end{pmatrix}\rightarrow\begin{bmatrix}1&-8&5&-13\\-3&2&-4&6\\13&2&12&k-12\end{bmatrix}\rightarrow\begin{pmatrix}3R1+R2\rightarrow R2\\-13R1+R3\rightarrow R3\end{pmatrix}\rightarrow\begin{bmatrix}1&-8&5&-13\\0&-22&11&-33\\0&106&-53&k+157\end{bmatrix}\rightarrow$$

$$\begin{pmatrix}-\frac{1}{22}R2\\\frac{1}{106}R3\end{pmatrix}\rightarrow\begin{bmatrix}1&-8&5&-13\\0&1&-\frac{1}{2}&\frac{3}{2}\\0&1&-\frac{1}{2}&\frac{k+157}{106}\end{bmatrix}\rightarrow\frac{3}{2}=\frac{k+157}{106}\rightarrow 3(53)=k+157\rightarrow k=2$$

$A-B-C=14-9-2=3$

Question 5

A = 20: Since $x>0$, $(x+1)^2<x^2+2x+4<(x+2)^2$ and $(2x)^2<4x^2+2x+1<(2x+1)^2$; so the number of integers N between these two radical expressions is $(2x+1)-(x+2)=x-1\rightarrow$ $2021^{2021}-1$. The powers of 21 end in 21, 41, 61, 81, 01, then repeat. 2021/5 = 404R1, so the exponential ends in 21, then after subtracting 1, we get 20.

B = 22: $\binom{2021}{1}(x)^{2020}(3)^1=6063x^{2020}\rightarrow 6063/8=757R7,\ 757/8=94R5,\ 94/8=11R6,$ $11/8=1R3\rightarrow 6063_{10}=13657_8\rightarrow 1+3+6+5+7=22$

$B-A=2$

Question 6

$A=-\frac{5}{13}$: Geometrically, this is a Quadrant I 5-12-13 triangle that gets moved to Quadrant IV. (The sine subtraction formula could also be used.)

$B=\frac{63}{16}$: $\tan(a-b)=\dfrac{\tan a-\tan b}{1-\tan a\tan b}=\dfrac{\frac{4}{13}+\frac{5}{12}}{1-\left(\frac{4}{13}\right)\left(\frac{5}{12}\right)}=\dfrac{63}{16}$

$C=-\frac{16}{65}$: $\sin(a+b)=\sin a\cos b+\cos a\sin b=\left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)+\left(\frac{4}{5}\right)\left(\frac{5}{13}\right)=-\frac{16}{65}$

$D=\frac{1}{2}$: $\tan\left(\text{Arctan}\,x+\text{Arctan}\frac{1}{3}\right)=\tan(\text{Arctan}\,1)=1\rightarrow\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}=$

$$\frac{\tan(\text{Arctan}\,x)+\tan\left(\text{Arctan}\frac{1}{3}\right)}{1-\tan(\text{Arctan}\,x)\tan\left(\text{Arctan}\frac{1}{3}\right)}=1=\frac{x+\frac{1}{3}}{1-\frac{x}{3}}=\frac{3x+3}{3-x}\rightarrow 3-x=3x+1\rightarrow x=\frac{1}{2}$$

$$\frac{100\left(\frac{63}{16}\right)\left(-\frac{16}{65}\right)\left(\frac{1}{2}\right)}{-\frac{5}{13}}=126$$

Question 7

A = 8: Let $x=\log_2 9$. $\left(\left(\log_2 9\right)^2\right)^{\frac{1}{\log_2(\log_2 9)}}\left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}\rightarrow\left((x)^2\right)^{\frac{1}{\log_2(x)}}\left(7^{\frac{1}{2}}\right)^{\log_7 4}\rightarrow\left(x^2\right)^{\log_x 2}(7)^{\log_7 4^{\frac{1}{2}}}\rightarrow$ $x^{\log_x 2^2}7^{\log_7 2}\rightarrow(4)(2)=8$

B= 488: The unpainted cubes comprise the $8\times8\times8$ inner cube which has 512 unit cubes. $1000-512=488$

$\frac{B}{A}=61$

Question 8

A = 6: The remainders can be 0, 1, 2, 3, 4, or 5, for a total of 6.

B = 46: $n=4$ works so all the numbers that satisfy the congruence relation must be of the form $n=4+5k$, where k is an integer. $0\le 4+5k\le 20 \to -4\le 5k\le 16 \to 0\le k\le 3 \to n=4,\ 9,\ 14,\ 19.$ $4+9+14+19=46$

C = 1: (We will end up using $13^2=169\equiv 4 \bmod 11$ and $4^5=1024\equiv 1 \bmod 11$.)

$$13^{20}=\left(13^2\right)^{10}\equiv 4^{10} \bmod 11=\left(4^5\right)^2 \bmod 11\equiv 1^2 \bmod 11=1 \bmod 11$$

D = 20: $2x+3\equiv -1 \bmod 4 \to 2x\equiv -4 \bmod 4\equiv 0 \bmod 4 \to x=2,\ 4,\ 6,\ 8.$ $2+4+6+8=20$

$A+B+C+D=73$

Question 9

A = 18: $2c=6\to c=3.$ $\frac{2a^2}{c}=12\to a^2=18.$ $a^2-b^2=c^2\to b^2=9.$ $(FW)^2=\left(\frac{2b^2}{a}\right)^2=\frac{4b^4}{a^2}=\frac{4(81)}{18}=18$

B = –1: $x^2+y^2+2x+4y-p=0\to(x+1)^2+(y+2)^2=p+5.$ The center is at (–1, –2) and the radius has length $\sqrt{p+5}$. Sketching this, we can see that the radius must be 2 and that the circle must cross the x-axis once and the y-axis twice, or the radius must be $\sqrt{1^2+2^2}=\sqrt{5}$ and cross the x-axis once, the y-axis once, and the origin. (There is no way to have three intersections where the circle cross the x-axis twice and the y-axis once.) So, finding p: $\sqrt{p+5}=2\to p=-1$ or $\sqrt{p+5}=\sqrt{5}\to p=0.$ $-1+0=-1$

$A+B=18+(-1)=17$

Question 10

A = 10: $x^3-5x^2+2x+5=2x+1\to x^3-5x^2+4=0.$ We can see here that the sum of the roots is 5, and we know that one root is 1. That means that $x_1+x_2=4.$ Instead of finding the other roots, just use the equation of the line that was given to us. $y_1+y_2=\left(2x_1+1\right)+\left(2x_2+1\right)=$ $2\left(x_1+x_2\right)+2=2(4)+2=10$

B = 15: Let $x+1$ be the missing factor. $x^4+4x^3+6px^2+4qx+r=\left(x+a\right)\left(x^3+3x^2+9x+3\right)=$

$$\left(x+a\right)\left(x^3\right)+\left(x+a\right)\left(3x^2\right)+\left(x+a\right)\left(9x\right)+\left(x+a\right)\left(3\right)=x^4+\left(a+3\right)x^3+\left(3a+9\right)x^2+\left(9a+3\right)x+3a$$

Now equate the corresponding parts. $a+3=4\to a=4;$ $3(1)+9=6p\to p=2;$ $9(1)+3=4q\to$ $q=3;$ $3(1)=r\to r=3\to(p+q)r=(2+3)(3)=15$

$A+B=25$

Question 11

There will be 18 payments to pay off the $18,000 balance.

Installment 1: $1000+\frac{1}{10}(18{,}000)=1000+1800$; Installment 2: $1000+\frac{1}{10}(17{,}000)=1000+1700$;

Installment 3: $1000+\frac{1}{10}(16{,}000)=1000+1600$, etc. The final installment will be $1000.

$$\sum_{n=1}^{18}\left(1000+\frac{1}{10}\left[18{,}000-(n-1)(1000)\right]\right)=\sum_{n=1}^{18}(2900-100n)=2900(18)-100\left[\frac{(18)(19)}{2}\right]=35{,}100$$

But don't forget about the initial $4000! The grand total will be $39,100.

Question 12

A = 81: $\left[f(x)\right]^2=\left[f(x-1)\right]\left[f(x+1)\right]\to\frac{f(x)}{f(x-1)}=\frac{f(x+1)}{f(x)}\to$ a geometric sequence

$$r=\pm\sqrt{\frac{9}{1}}=\pm3\to f(5)=9\left(\pm3\right)^2=81$$

B = −18: $2g(x)=g(x)+g(x)=g(x+1)+g(x-1)\to g(x)-g(x-1)=g(x+1)-g(x)\to$ an arithmetic sequence. $d=\frac{g(6)-g(2)}{4}=\frac{2-6}{4}=-1$. $g(26)=g(6)+20(-1)=-18$

C = 144: $h(x+1)h(x-1)=\left[h(x)\right]^2-\left(-1\right)^x\to h(0)=1,\ h(1)=1,\ h(2)h(0)=\left[h(1)\right]^2-\left(-1\right)^1\to$ $h(2)=1+1=2,\ h(3)h(1)=\left[h(2)\right]^2-\left(-1\right)^2=4-1=3,\ h(4)h(2)=\left[h(2=3)\right]^2-\left(-1\right)^3\to$ $2h(4)=9+1\to h(4)=5$. We can definitely tell this is the Fibonacci sequence since we just got 1, 1, 2, 3, 5. Counting up to $h(11)$ we have 8, 13, 21, 34, 55, 89, then 144.

$$\cos\left[5\left(B+C-A\right)\right]^\circ=\cos\left[5\left(-18+144-81\right)\right]^\circ=\cos\left[5(45)\right]^\circ=\cos225^\circ=-\frac{\sqrt{2}}{2}$$

Question 13

A = 8: The dot product of the direction vector of the line and the normal vector of the plane is 0: $4(2)+4(3)-k(4)=0 \to k=5.$ To find d, plus in the point that the line goes through: $4(3)+4(4)-(5)(5)=d=3.$ $5+3=8$

B = 4: We can tell that the point (2, −1, 2) that is given to us in the equation of the line is contained on the plane because these x-, y-, and z-values sum to 5. Therefore, that is the point of intersection of the line and the plane. So the distance from point to point is

$$\sqrt{3^2+4^2+12^2}=13 \to \left\lfloor \frac{13}{3} \right\rfloor = 4$$

C = −3: For these two lines to be coplanar, the determinant of the matrix formed by the coordinates from the three direction vectors must be 0. Two of these vectors are the denominators of the symmetric form of the line, and the other vector is the displacement vector between the two points: $\langle 1-2,\, 4-3,\, 5-4 \rangle = \langle -1,\, 1,\, 1 \rangle$.

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0 = (-1)(1+2k)-(1)\left(1+k^2\right)+(1)(2-k)=k^2+3k.$$ The sum is −3.

$$\frac{B^{-C}}{A}=\frac{4^{-(-3)}}{8}=\frac{64}{8}=8$$

Question 14

$$A=\frac{321}{500}:\ \left(\frac{17}{20}\right)\left(\frac{3}{5}\right)+\left(\frac{33}{100}\right)\left(\frac{2}{5}\right)\to\frac{51}{100}+\frac{66}{500}=\frac{321}{500}$$

$$B=\frac{85}{107}:\ \frac{\left(\frac{17}{20}\right)\left(\frac{3}{5}\right)}{\frac{321}{500}}=\left(\frac{51}{100}\right)\left(\frac{500}{321}\right)=\frac{255}{321}=\frac{85}{107}$$

$$AB=\left(\frac{321}{500}\right)\left(\frac{85}{107}\right)=\frac{51}{100}\to 51(\%)$$