2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

Answer Key: ACBAC ADADB DDCAA AEBAC ACCBD BDCBB

1.) A, since P(x) has only one real root then we know that the value of its discriminant must be equal to 0. Thus, $16k^2 - 4(3)(7) = 0$. $k^2 = \frac{4(3)(7)}{16} = \frac{21}{4}$ so for $k > 0$, $k = \frac{\sqrt{21}}{2}$. $m + n = 23$.

2.) C, to convert from rectangular form to polar from let x = $r\cos\theta$ and $y = r\sin\theta$, and

$x^2 + y^2 = r^2$. $x^2 + y^2 = 2y$ gives $r^2 = 2r\sin\theta$. Divide by r to get $r = 2\sin\theta$.

3.) B, firstly put the x parametric equation into terms of t and then substitute in the other equation. $t = 2(x-1)$ and $y = 4(x-1)^2 - 1$ which has a parabolic graph with vertex (1, -1). AB = -1.

4.) A, to solve these types of inequalities split it up into two inequalities by having the first set of inequality and the second set of inequality. Doing such results with $x + 1 \le 2x + 3$ and $2x + 3 \le -3x + 8$. Now simply solve each of them separately then combine the results. Thus,

$x + 1 \le 2x + 3$

$1 \le x + 3$

$-2 \le x$

$x \ge -2$

$2x + 3 \le -3x + 8$

$5x + 3 \le 8$

$5x \le 5$

$x \le 1$

Combining these two inequalities we get $-2 \le x \le 1$. Thus, the integral solutions are -2, -1, 0, and 1 for a total of 4 integral solutions.

5.) C. Let a and b be the two numbers. Thus,

$\sqrt{a^2 + b^2} = 8$

$a^2 + b^2 = 64$

$\frac{a+b}{2} = 5$

$a + b = 10$

$(a + b)^2 = a^2 + 2ab + b^2$

$(10)^2 = a^2 + b^2 + 2ab$

$100 = 64 + 2ab$

$36 = 2ab$

$ab = 18$

6.) A, firstly we need to solve the equation and in order to do so we must first square both sides. Thus,

$\sqrt{x+1} = \sqrt{2x-5}+1$

$(\sqrt{x+1})^2 = (\sqrt{2x-5}+1)^2$

$x+1 = 2x-5+2\sqrt{2x-5}+1$

$x+1 = 2x-4+2\sqrt{2x-5}$

$-x+5 = 2\sqrt{2x-5}$

$\frac{-x+5}{2} = 2\sqrt{2x-5}$

$\left(\frac{-x+5}{2}\right)^2 = \left(\sqrt{2x-5}\right)^2$

$\frac{x^2-10x+25}{4} = 2x-5$

$x^2-10x+25 = 8x-20$

$x^2+18x+45 = 0$

$(x-3)(x-15) = 0$

$x = 3, x = 15$

After plugging in both solutions, we find that only x = 3 works and therefore x = 15 is extraneous. Now, since 3 is the area of circle P we need to find its radius.

$A = \pi r^2$

$3 = \pi r^2$

$\frac{3}{\pi} = r^2$

$\sqrt{\frac{3}{\pi}} = r$

$\sqrt{\frac{3}{\pi}} = \frac{\sqrt{3\pi}}{\pi}$

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

7.) D, to find the vertex of a parabola we can use $\left(\frac{-b}{2a}, Q\left(\frac{-b}{2a}\right)\right)$. The ordinate value is the y-value. Thus,

$$\frac{-b}{2a} = \frac{-(-5)}{6} = \frac{5}{6}$$

$$Q\left(\frac{5}{6}\right) = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) + 11$$

$$Q\left(\frac{5}{6}\right) = \frac{25}{12} - \frac{50}{12} + \frac{132}{12}$$

$$Q\left(\frac{5}{6}\right) = \frac{107}{12}$$

8.) A, in order to solve absolute value inequalities firstly break up the inequality to two pieces then combine it later. Specifically, we are testing the positive and negative cases and to test the positive case we leave the expression within the absolute value as is and to test the negative case we multiply a -1 to each term within the absolute value. Thus,

$$|2x - 3| < 5$$

Positive Case	Negative Case
$2x - 3 < 5$	$-2x + 3 < 5$
$2x < 8$	$-2x < 2$
$x < 4$	$x > -1$

Combining these two inequalities we get $-1 < x < 4$ and therefore there are 4 integral solutions, namely 0,1,2, and 3.

9.) D,

$$\sec^4 x - 6\tan^2 x + 2 = 0$$

$$(\tan^2 x + 1)^2 - 6\tan^2 x + 2 = 0$$

$$\tan^4 x - 4\tan^2 x + 3 = 0$$

$$\tan^2 x = 1 \qquad \tan^2 x = 3$$

$$\tan x = \pm 1 \qquad \tan x = \pm\sqrt{3}$$

The solutions to this are x $\in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$. The sum of these is 8π.

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

10.) B,

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\sqrt{||u||}\sqrt{||v||}}$$

$$\cos\theta = \frac{(-3)(1)+(5)(2)+(2)(4)}{\sqrt{(-3)^2+\ (5)^2+\ (2)^2}\sqrt{(1)^2+\ (2)^2+\ (4)^2}}$$

$$\cos\theta = \frac{-3+10+8}{\sqrt{38}\sqrt{21}}$$

$$\cos\theta = \frac{15}{\sqrt{798}}$$

$$\cos\theta = \frac{15\sqrt{798}}{798}$$

$$\cos\theta = \frac{5\sqrt{798}}{266}$$

$$\theta = \cos^{-1}\frac{5\sqrt{798}}{266}$$

11.) D, in order to solve these kinds of composite functions work from the inside out step-by-step. Thus,

$$g(4) = \frac{1}{(4)-5} = \frac{1}{-1} = -1$$

$$g(-1) = \frac{1}{(-1)-5} = \frac{1}{-6} = -\frac{1}{6}$$

$$f\left(-\frac{1}{6}\right) = (-\frac{1}{6})^2 - 5\left(-\frac{1}{6}\right) + 1 = \frac{1}{36} + \frac{5}{6} + 1 = \frac{1}{36} + \frac{30}{36} + \frac{36}{36} = \frac{67}{36}$$

12.) D, Since we know that the length of the latus rectum of a parabola is 4p we know that the distance between the focus point and the parabola is 2p. Since we know that p =1 we know that this length is 2. Additionally, we know that the shortest distance will also be 2 is this is the definition of a parabola which is a locus of points equidistant from a point (the focus) and a line (the directrix). Realizing this will result in a right triangle with both legs being 2. Solving this with area = $\frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$.

13.) C, Let x be negative. Then $|y| - \frac{2x}{|x|} = |y| + 2 = -1$ and $|y| = -3$, which is impossible. Thus, x is positive and $|y| - 2 = -1$ so $|y| = 1$. If $y = 1$, then $x|x| + 1 = 24$, or $x^2 = 23$. If $y = -1$, then $x|x| - 1 = 24$, or $x^2 = 25$. The sum of the possible values of x^2 is $23 + 25 = 48$.

14. **C.** $\frac{\frac{x-1}{x+1} \div \frac{x-1}{x}}{\frac{x}{x+1} - \frac{1}{x-1}} = \left(\frac{x-1}{x+1} \div \frac{x-1}{x}\right) \div \left(\frac{x(x-1)-1(x+1)}{(x-1)(x+1)}\right) = \frac{x}{x+1} \div \frac{x^2-2x-1}{(x-1)(x+1)} =$

$\frac{x}{x+1}\frac{(x-1)(x+1)}{x^2-2x-1} = \quad \frac{x(x-1)}{x^2-2x-1}$

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

15.) A, firstly we need to factor $2x^2 - 11x - 40$.

$2x^2 - 11x - 40 = (2x+5)(x-8)$

$(2x+5)(x-8) = 0$

$x = 8, x = \dfrac{-5}{2}$

With these values we test values around them to check on which direction has the positive values namely ≥ 0. We find that as we plug in values for large x (larger than 8) we get a positive value and when we plug in for small x (less than $\frac{-5}{2}$) we also get a positive value. Thus, our solution is

$\left(-\infty, \frac{-5}{2}\right] \cup [8, \infty)$

16.) A. $2\,sin\,\theta\,cos\,\theta = cos\,\theta$. $2\,sin\,\theta\,cos\,\theta - cos\,\theta = 0$. $cos\,\theta\,(2\,sin\,\theta - 1) = 0$. $cos\,\theta = 0$ or

$sin\,\theta = \frac{1}{2}$. Reference angles over $[0, 2\pi]$ are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$. Solutions are $\theta = 2\pi n + \frac{\pi}{6}, 2\pi n + \frac{5\pi}{6}, \pi n + \frac{\pi}{2}$

17.) E, to evaluate limits as they approach ∞ we need to evaluate the rate in which the numerator is increasing and the rate in which the denominator is increasing. To do this we determine the highest degree within both the numerator and the denominator. In our case, the highest degree in the numerator is the term $-x^7$ whilst in the denominator the highest term is $2x^4$. Comparing the two, it is clear that $-x^7$ is increasing faster in the negative direction when compared to $2x^4$ since 7 > 4. Thus,

$$\lim_{x\to\infty} \frac{x^4 - 6x^3 + x^2 - x^7 + x - 1}{2x^4 + x^3 - x^2 - 6x + 5} = -\infty$$

18.)B, the eccentricity of any non-degenerate conic section is e = $\frac{c}{a}$. We must first put the conic in standard form. Thus,

$2x^2 + y^2 + 12x + 4y - 22 = 0$

$2(x^2 + 6x) + (y^2 + 4y) = 22$

$2(x^2 + 6x + 9) + (y^2 + 4y + 4) = 22 + (2)(9) + (1)(4)$

$2(x+3)^2 + (y+2)^2 = 44$

$\dfrac{(x+3)^2}{22} + \dfrac{(y+2)^2}{44} = 1$

$c^2 = a^2 - b^2$

$c^2 = 44 - 22$

$c = \sqrt{22}$

$a = \sqrt{44}$

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

$$e = \frac{\sqrt{22}}{\sqrt{44}}$$

$$e = \frac{\sqrt{(22)(22)(2)}}{44}$$

$$e = \frac{22\sqrt{2}}{44}$$

$$e = \frac{\sqrt{2}}{2}$$

Thus, 2+2 = 4 which is the units digit.

19.) A, using this information we can plug it into the general form of a circle which is

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x+4)^2 + (y-9)^2 = 16$$

$$x^2 + 8x + 16 + y^2 - 18y + 81 = 16$$

$$x^2 + y^2 + 8x - 18y + 81 = 0$$

1+1+8-18+81 = 73 the units digit of which is 3.

20.) C. $\dfrac{1-\cos^2 x}{\sin x\left(\dfrac{\cos x}{\sin x}\right)}$ $\frac{1+cos\,x}{sin\,x}\frac{1-cos\,x}{cot\,x} = \frac{sin^2\,x}{cos\,x}$

21.) A, $\lim_{h\to 0} \frac{(x^2+2xh+h^2+6x+6h+9)-(x^2+6x+9)}{h}$

$$\lim_{h\to 0} \frac{2xh + h^2 + 6h}{h}$$

$$\lim_{h\to 0} \frac{h(2x+h+6)}{h}$$

$$\lim_{h\to 0} 2x + h + 6$$

$$2x + 6$$

Alternative Solution:

This is first principles which is to find the derivative of $f(x)$. Thus,

$$f(x) = (x+3)^2$$

$$f'(x) = (2)(1)(x+3)^{2-1}$$

$$f'(x) = 2x + 6$$

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

22.) C, to find the inverse of a function switch x and y then simplify in terms of y.

$$y = \frac{x+1}{x-1}$$

$$x = \frac{y+1}{y-1}$$

$$xy - x = y + 1$$

$$-x = y - xy + 1$$

$$-x - 1 = y(1 - x)$$

$$\frac{-x-1}{1-x} = \frac{x+1}{x-1} = y$$

23.) C, to solve these kinds of questions it is useful to get an idea of what the graph looks like. To do this set one of the pairs of absolute value expression equal to 0 by plugging in the appropriate value for x or y. Then solve for the other variable. Thus,

$$2x = 0$$

$$x = 0$$

$$|2(0)| + |y - 1| = 8$$

$$|y - 1| = 8$$

$$y = -7, y = 9$$

The graph has points (0,-7) and (0,9).

$$y - 1 = 0$$

$$y = 1$$

$$|2x| + |(1) - 1| = 8$$

$$|2x| = 8$$

$$x = 4, x = -4$$

The graph has points (1,4) and (1,-4).

By realizing this graph is composed of absolute value function we realize that it is some form of a quadrilateral, specifically it is a kite. The area of which can be found by $\frac{1}{2}(diagonals)$. To solve for both of our diagonals is simply the distance between (1,4) and (1,-4) and for the other diagonal it is (0,-7) and (0,9). Thus, the lengths of the diagonals being 8 and 16 respectively. The area is (8)(16)$\left(\frac{1}{2}\right) = 64$.

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

24.) B, to solve these kinds of questions you will have to take each trig function or inverse trig function step by step. Since we know that an inverse trig function and its trig function cancel each other out we use this to solve our equation. Thus,

$\sin(\cos^{-1}(\tan(\cos^{-1}(\tan x)))) = 1$

$\cos^{-1}(\tan(\cos^{-1}(\tan x))) = \sin^{-1}(1)$

$\cos^{-1}(\tan(\cos^{-1}(\tan x))) = \frac{\pi}{2}$

$\tan(\cos^{-1}(\tan x)) = \cos\frac{\pi}{2}$

$\tan(\cos^{-1}(\tan x)) = 0$

$\cos^{-1}(\tan x) = \tan^{-1} 0$

$\cos^{-1}(\tan x) = 0$

$\tan x = \cos 0$

$\tan x = 1$

$x = \tan^{-1} 1$

$x = \frac{\pi}{4}$

25.) D, The floor of each variable must either be 1, 2, or 3, and their product must be at most 3, leaving only a couple cases for the unordered triplet $(\lfloor a\rfloor, \lfloor b\rfloor, \lfloor c\rfloor)$: $(1,1,1)$, $(1,1,2)$, and $(1,1,3)$. By dividing each equation by its floor function and multiplying all three equations together, we obtain $abc = \sqrt{\frac{60}{\lfloor a\rfloor\lfloor b\rfloor\lfloor c\rfloor}}$. Note that $abc \geq ab\lfloor c\rfloor = 5$, so $\sqrt{\frac{60}{\lfloor a\rfloor\lfloor b\rfloor\lfloor c\rfloor}} \geq 5$ and $\lfloor a\rfloor\lfloor b\rfloor\lfloor c\rfloor \leq 2.4$, eliminating the third unordered triple.

For the triple $(1,1,1)$, $bc = 3$ and $abc = \sqrt{60}$, so $a = \frac{\sqrt{60}}{3}$. However, this is greater than 2 so $\lfloor a\rfloor \neq 1$ and it is rejected.

For the triple $(1,1,2)$, $ab = \frac{5}{2}$ so the solution is $\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5}\right)$, which is valid.

For the triple $(1,2,1)$, $ac = 2$ so the solution is $\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5}\right)$, which is valid.

For the triple $(2,1,1)$, $bc = \frac{3}{2}$ so $a = \frac{2\sqrt{30}}{3}$. However, this is greater than 3 so $\lfloor a\rfloor \neq 2$ and it is rejected.

The two possible values of b are $\frac{\sqrt{30}}{4}$ and $\frac{\sqrt{30}}{2}$, which sum to $\frac{3\sqrt{30}}{4}$.

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

26.) B,

I is false if one of the lines is vertical and the other horizontal, where the product of the slopes does not exist.

II is true as since the line is tangent to the circle at the intersection the radius creates 2 right angles.

III is false as $f(x) = \frac{x^2}{x} = x$ with a hole of discontinuity at $x = 0$. There are no asymptotes.

27.) D, to find the derivative of a quotient we use: $\frac{(B)(T')-(T)(B')}{B^2}$ where B refers to the denominator and T refers to the numerator. Thus,

$$g(x) = \frac{x-2}{2x+1}$$

$$g'(x) = \frac{(2x+1)(1)-(x-2)(2)}{(2x+1)^2}$$

$$g'(2) = \frac{(2(2)+1)(1)-((2)-2)(2)}{(2(2)+1)^2}$$

$$g'(2) = \frac{5-0}{5^2}$$

$$g'(2) = \frac{1}{5}$$

28.) C, using the half angle formula we get $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$. Since we already have cosx we need to evaluate sinx by constructing a right triangle with a leg of a and a hypotenuse of b. Thus,

$$b^2 = a^2 + c^2$$

$$b^2 - a^2 = c^2$$

$$\sqrt{b^2-a^2} = c$$

$\sin x = \frac{c}{b} = \frac{-\sqrt{b^2-a^2}}{b}$ (Note: this is negative due to position)

$$\tan\frac{x}{2} = \frac{\frac{-\sqrt{b^2-a^2}}{b}}{1+\frac{a}{b}}$$

$$\tan\frac{x}{2} = \frac{-\frac{\sqrt{b^2-a^2}}{b}}{\frac{a+b}{b}}$$

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

29.) B, in order to solve for x we must utilize the fact that $\cosh^2 x - \sinh^2 x = 1$ Thus,

$$\cosh^2 x - \left(\frac{2}{3}\right)^2 = 1$$

$$\cosh^2 x = 1 + \frac{4}{9}$$

$$\cosh^2 x = \frac{13}{9}$$

$\cosh x = \frac{\sqrt{13}}{3}$ as the cosh function is always positive.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

$$\cosh x + \sinh x = \frac{e^x + e^{-x} + e^x - e^{-x}}{2}$$

$$\cosh x + \sinh x = \frac{2e^x}{2}$$

$$\cosh x + \sinh x = e^x$$

$\frac{\sqrt{13}}{3} + \frac{2}{3} = e^x$ (by substitution)

$$x = \ln\frac{2+\sqrt{13}}{3}$$

2 + 13 + 3 = 18

2022 Alpha Equations and Inequalities – Washington, DC

Answers and Solutions

30.) B,

$2\cosh 2x + 6\sinh 2x = 12$

$2\left(\frac{e^{2x}+e^{-2x}}{2}\right) + 6\left(\frac{e^{2x}-e^{-2x}}{2}\right) = 12$

$e^{2x} + e^{-2x} + 3e^{2x} - 3e^{-2x} = 12$

$4e^{2x} - 2e^{-2x} = 12$

$2e^{2x} - e^{-2x} = 6$

$2e^{2x} - 6 - e^{-2x} = 0$

$2e^{4x} - 6e^{2x} - 1 = 0$

Let $u = e^{2x}$. Thus,

$2u^2 - 6u - 1 = 0$

$u = \frac{-(-6) \pm \sqrt{6^2 - (4)(2)(-1)}}{2(2)}$

$u = \frac{6 \pm \sqrt{44}}{4}$

$u = \frac{6 \pm 2\sqrt{11}}{4}$

$u = \frac{3 \pm \sqrt{11}}{2}$

We only use positive case as u must be positive

$u = e^{2x}$

$\frac{3+\sqrt{11}}{2} = e^{2x}$

$\ln\left(\frac{3+\sqrt{11}}{2}\right) = 2x$

$x = \frac{1}{2}\ln\left(\frac{3+\sqrt{11}}{2}\right)$

Thus, 2 + 3 + 11 + 2 = 18