Alpha Individual Test Answers and Solutions

1. E
2. D
3. B
4. C
5. D
6. A
7. B
8. A
9. C
10. A
11. B
12. C
13. C
14. C
15. C
16. A
17. B
18. E
19. C
20. A
21. A
22. C
23. B
24. C
25. B
26. B
27. D
28. C
29. A
30. E

1. E To solve for x on $[0, 2\pi)$, we will need to solve for $2x$ on $[0, 4\pi)$. Sine is positive in quadrants I and II, so if we let $a = \sin^{-1}\frac{2}{3}$, we have $2x = a, \pi - a, 2\pi + a, 3\pi - a$. The sum of the values of $2x$ is 6π, so the sum of the solutions is 3π.

2. D Converting the quantities to polar, $\left(2\operatorname{cis}\frac{\pi}{3}\right)^5\left(2\sqrt{2}\operatorname{cis}\frac{-\pi}{4}\right)^6 = \left(2^5\operatorname{cis}\frac{-\pi}{3}\right)\left(2^9\operatorname{cis}\frac{\pi}{2}\right)$ So the final answer is $2^{14}\operatorname{cis}\frac{\pi}{6}$

3. B Left multiply by $\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}^{-1}$ on both sides, $M = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 39 \\ -1 & -16 \end{bmatrix}$

4. C For $f(x)$ to be defined, we need $x + 2 \geq 0$ and $x - 2 > 0$. Thus $x > 2$.

5. D $\frac{4}{\cos^2 x} + \frac{4}{\sin^2 x} = \frac{4\sin^2 x + 4\cos^2 x}{\cos^2 x \sin^2 x} = \frac{4}{\cos^2 x \sin^2 x} = \frac{16}{(2\cos x \sin x)^2} = 16\csc^2 2x$

6. A The sum of roots is 4, and the sum of roots taken 2 at a time is 7, so the sum of the squares of the roots is $4^2 - 2 \cdot 7 = 2$.

7. B The scalar triple product is equivalent to the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, which is 0.

8. A Moving logs to the same side and combining, we have $\log_2\frac{(3x-4)}{\sqrt{x}} = 2$. Removing the logs, $\frac{3x-4}{\sqrt{x}} = 2^2$, or $3x - 4\sqrt{x} - 4 = 0$. $\left(3\sqrt{x} + 2\right)\left(\sqrt{x} - 2\right) = 0$. $\sqrt{x} = -\frac{2}{3}, 2$, but $\sqrt{x}$ cannot be negative, so $x = 4$. It can be verified that 4 is not extraneous. So the sum of the solutions is $\frac{4}{1}$, and $m + n = 5$.

9. C There are $\binom{15}{3} = 455$ ways to draw 3 marbles out of the bag. We need one of each color, so the number of ways to do so is $\binom{4}{1}\binom{6}{1}\binom{5}{1} = 120$, so the probability is $\frac{120}{455} = \frac{24}{91}$.

10. A The constant is less than the coefficient on $\cos\theta$, so r can go negative, thus it is a looped limacon.

11. B 120° is the largest angle in the triangle, so it must face the longest side. $3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos 120° = 9 + 25 + 15 = 49$, so the longest side is 7.

12. C In base 6, $0.\overline{25} = \frac{25}{55}$, converting 25 and 55 to base 10, we have $\frac{17}{35}$.

13. C Let a, b be the lengths of the two legs, we have $a^2 + b^2 = 64$ and $ab = 16$. We are looking for the tangent of the smallest angle, which is $\frac{a}{b}$ or $\frac{b}{a}$, whichever is less than 1. From the given information, we have $a^2 + b^2 = 4ab$. We can then divide both sides by ab to get $\frac{a}{b} + \frac{b}{a} = 4$. Let $x = \frac{a}{b}$, we have $x + \frac{1}{x} = 4$, or $x^2 - 4x + 1 = 0$. Now $(x-2)^2 = 3$, so $x = 2 - \sqrt{3}$, since we are taking the value less than 1.

14. C There are vertical asymptotes at $x = \pm 2$. Asymptotically, both numerator and denominator are linear. As x approaches ∞, both are positive, so $y = 1$ is a horizontal asymptote. As x approaches $-\infty$, the numerator is negative and the denominator is positive, so $y = -1$ is also a horizontal asymptote. In total, there are 4 asymptotes.

15. C Note that $123456 - 12345 = 111111$, which is a multiple of 11. So as long as $n - 54321$ is also a multiple of 11, the result must be a multiple of 11. $n = 54321$ certainly satisfies that.

16. A The right hand side is a sinusoid of amplitude 10. The shift is irrelevant for the area, as it just rotates the graph. $r = 10\cos\theta$ is a circle with diameter 10, so the area is 25π.

17. B Since $8x + 5y$ has no z term in it, we can attempt to eliminate z from the initial system. Multiplying the first equation by -1 and second by 2, then adding them together, we have $8x + 5y = 22$. 11 divides into 22.

18. E While the problem is laid out identically to 17, eliminating z does not help, so we need to solve for (x, y) in terms of z. Note that $\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1$, so for any integer value of z, there will be an ordered pair of integers for (x, y). We can rewrite the system as $\begin{cases} 2x + y = 28 - 2z \\ 5x + 3y = 25 - z \end{cases}$. Solving this system, we have $x = 59 - 5z$, $y = 8z - 90$. So $5x + 8y = 39z - 425$. 39 and 425 do not share a prime factor, so there is not a prime that is guaranteed to divide into $39z - 425$.

19. C Let R be the radius of the larger circle, then the annulus has area $\pi(R^2 - 25) = 25\pi$. The larger radius is the hypotenuse of a right triangle with a smaller radius and half of the chord as legs, so half of the chord must also be of length 5. Therefore, the entire chord has length 10.

20. A For the graph to be two intersecting lines, the equation must factor into the form of $(ax + by + c)(mx + ny + p) = 0$. Based on the three quadratic terms, we have $am = 6, bn = 6, an + bm = 13$.
There are three equations and four variables, but the two factors are arbitrarily scalable, so we can simply designate $a = 2, m = 3$. From there, we have $bn = 6$ and $2n + 3b = -13$, from which we can find either $b = -3, n = -2$ or $b = -\frac{4}{3}, n = -\frac{9}{2}$. Note that the two solutions are just different scalings of the same solution, as $\frac{a}{b}$ and $\frac{m}{n}$ are some arrangements of $-\frac{2}{3}$ and $-\frac{3}{2}$ in both cases. For convenience, we will take the integer solution.
We will next use the linear terms. We have $2p + 3c = -4$ and $-3p - 2c = 1$. Solving this system, we have $c = -2, p - 1$. We can verify that $cp = -2$ matches our original equation.

The two lines are $2x - 3y - 2 = 0$ and $3x - 2y + 1 = 0$, we are looking for $x + y$ for the solution pair (x, y). It is sufficient to subtract the first equation from the second, to get $x + y + 3 = 0$, so $x + y = -3$. Alternatively, simply solve the system to get $x = -\frac{7}{5}, y = -\frac{8}{5}$.

21. A We have already found the equations of the two lines in question 20, so we will pick things up from there. The slopes of the two lines are $\frac{3}{2}$ and $\frac{2}{3}$. We can use difference of angles to find the tangent of the angle between the lines. As $\frac{\frac{3}{2}-\frac{2}{3}}{1+\frac{2}{3}\cdot\frac{3}{2}} = \frac{5}{12}$. Thus sine of the angle is $\frac{5}{13}$.

22. C The gear on the wheel spins at twice the rate as the gear on the pedals, so the angular velocity of the wheel is 2 revolutions per second. From here, it's a unit analysis problem: $\frac{2\,rev}{s} \cdot \frac{3600s}{1hr} \cdot \frac{200cm}{1rev} \cdot \frac{1km}{10^5cm} = \frac{72km}{5hr} = 14.4km/hr.$

23. B Note that there are 2 separate E's, so they would be combined. Let h, e, l, n, a be the exponent on the corresponding letters, we have $h + e + l + n + a = 6$ where all variables are nonnegative integers. This is stars and bars with 6 stars and 4 bars, so the number of terms is $\binom{10}{4} = 210$

24. C 11 minutes after Milaan started, Milaan has walked 3300 feet, Kasra jogged 4200 feet, Alex has run for 5400 feet, and Bailey has sprinted for 7200 feet. If the length of the track is t, then when t is divided into each of the distances, the remainder must be the same. So t must divide into the differences of all the distances, which are 900, 1200, and 1800. The greatest common divisor of the three is 300 feet, which is the maximum possible length.

25. B Let the number of cards Amy and Eddie can get done each hour by themselves be a and e respectively, then we have $\frac{10}{3}a = 4e = \frac{5}{2}(a + e - 18)$. From the first equality, we have $e = \frac{5}{6}a$, so $\frac{10}{3}a = \frac{5}{2}\left(a + \frac{5}{6}a - 18\right)$, or $a = 36$. Thus the number of cards in the set is $\frac{10}{3} \cdot 36 = 120$.

26. B ΔALI is isosceles with $m\angle ALI = 108°$. The minimum possible length for KL occurs when $KIRA$ is a rectangle and K is on the same side of AI as L. Let M be the midpoint of segment AI. When $KIRA$ is a square, $KM = AM = 20$. $\frac{LM}{AM} = \tan 36°$. Using the approximation provided in the problem, $\tan 36° = \frac{3}{4}$, so $LM = 15$. Therefore $KL = 20 - 15 = 5$.

27. D Initially, Jack's time exceeds Andrew's time quickly, and their times will not match after $T = 2$ for quite a while. It is relatively easy to determine the last value of T when their times match up. It is after they have each made 29 moves, as the average time per move for Andrew is also 15 seconds per move. Starting from the 30th move, Andrew's time start to greatly exceed Jack's time. We can work our way back from there. The times listed below are the time spent at the end of each move.

Move number	29	28	27	26
Andrew's time	435	406	378	351
Jack's time	435	420	405	390

Based on this, after $T = 2$, Andrew and Jack's times match again when they have each spent 405, 406, 420, and 435 seconds on their moves, for a total of 5 times.

28. C Imagine if the rabbit's playpen is mirrored to the other side of the wall, then the resulting shape is a hexagon with perimeter of 24 feet. The largest possible area occurs when the hexagon is regular, so the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} \cdot 4^2 = 24\sqrt{3}$. The desired area of the quadrilateral, which is an isosceles trapezoid is $12\sqrt{3}$.

29. A Of the two restrictions, the one on F's is more restrictive, as any 3 spots can be chosen for E!E, so long as ! occupies the middle spot. For the F's, the parity of their positions must be opposites, so there are $4 \cdot 4$ choices for where they will go. Of the remaining 6 spots, 3 will be taken for E!E. There are $\binom{6}{3}$ ways to do that. Finally, JRY can be placed in any remaining position with no restrictions. There are 3! ways to place those letters. Putting it all together, there are $4 \cdot 4 \cdot \binom{6}{3} \cdot 3! = 16 \cdot 20 \cdot 6 = 1920$ valid arrangements.
As a side note, this problem is basically the arrangements of the back row in Fischer random, also known as chess 960 for the 960 possible arrangements. In Fischer random, the back rows are randomly arranged under the two restriction – the Bishops must be on opposite colors, and the King must be between the Rooks. The difference in this problem is that there are essentially two distinct Knights.

30. E For the random number generators, we will consider an outcome of 0 to be a miss, and outcome of 5 or 10 to be a hit. For Eric to produce a sum of 10, he either needs 2 hits from the first random number generator, or at least 1 hit from the second random number generator. In essence, we need 2 hits from the first random number generator and 2 misses from the second random number generator to be complementary.
2 hits from the first random number generator has probability of p^2.
2 misses from the second random number generator has probability of $(1.3 - p)^2$.
Combining them, we have $p^2 + (1.3 - p)^2 = 1$, or $2p^2 - 2.6p + 0.69 = 0$.
We need an approximation for p to the nearest 0.05 or so, as the answer choices are 0.1 apart. We can do this by completing the square here. We have

$$p^2 - 1.3p + 0.65^2 = -0.345 + 0.65^2$$
$$(p - 0.65)^2 = 0.0775$$
$$p - 0.65 = \pm\sqrt{0.0775}$$

$0.3^2 = 0.09$, so $\sqrt{0.0775}$ is slightly less than 0.3. Additionally, $p \geq 0.5$, so $p \approx 0.65 + 0.3 = 0.95$, so 0.94 is the closest choice. ($0.28^2 = 0.0784$ would have been a better approximation, but it is not necessary. The actual value for p accurate to 3 places is 0.928.)