All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

∼∼∼∼∼∼∼∼ *Good luck, and have fun!* ∼∼∼∼∼∼∼∼

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1) Happy 2022! (That's an exclamation mark, not a factorial.) Find $\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix}$.

A) 0 B) 2 C) 4 D) 2022 E) NOTA

2) If $\begin{bmatrix} 3 & 5 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, find xy.

A) $-\frac{1}{4}$ B) $-\frac{1}{8}$ C) $\frac{1}{8}$ D) $\frac{1}{4}$ E) NOTA

3) If $M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 7 \\ 1 & 8 \\ 2 & 8 \end{bmatrix}$, which of the following can exist?

A) $MM^T + N^T N$ B) $(NM)^2$ C) $M + N^T$
D) e^M E) NOTA

4) For the matrix $\begin{bmatrix} b & 2a \\ 2c & b \end{bmatrix}$, $b^2 - 4ac$ is which of the following?

A) Determinant (only) B) Discriminant (only) C) Both A) and B)
D) Neither A) nor B) E) NOTA

5) Which of the following is not a unit vector?

A) $\frac{\hat{\mathbf{i}}}{3} + \frac{\hat{\mathbf{j}}}{4} + \frac{\hat{\mathbf{k}}}{5}$ B) $\hat{\theta}$
C) $\langle -1 \rangle$ D) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4} \right\rangle$
E) NOTA

6) $\begin{vmatrix} x & 2 & -y \\ -3 & 1 & 4 \\ y & 0 & x \end{vmatrix} = 11$ defines an ellipse. Find its area.

A) 11π B) 16π C) 18π D) 25π E) NOTA

7) A set of points defines a shape with area 6. Each point (x_i, y_i) in the shape is shifted to the point $(3x_i + 2y_i, 5x_i + 6y_i)$. Find the new area of the shape they define.

A) 48 B) 60 C) 108 D) $6\sqrt{533}$ E) NOTA

8) A is a 5-by-5 matrix such that $|A| = \sqrt[5]{2}$. Find $|2A|$.

A) 2 B) $2\sqrt[5]{2}$ C) 32 D) $32\sqrt[5]{2}$ E) NOTA

9) When solving a system of equations (whose three variables are x, y, and z) with Cramer's Rule, Romir obtains the following.

$$x = \frac{\begin{vmatrix} 6 & -2 & -1 \\ -8 & 4 & 3 \\ 7 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -2 & -1 \\ -2 & 4 & 3 \\ 5 & -1 & 0 \end{vmatrix}}$$

Find $x + y + z$.

A) -1 B) $-\frac{1}{3}$ C) $\frac{1}{3}$ D) 1 E) NOTA

10) $\mathbf{u}$ and $\mathbf{v}$ are vectors of length 3 and 5 respectively such that the following conditions are met: both of their tails are on the origin, and $||\mathbf{v} - \mathbf{u}|| = 7$. Find the smaller angle formed by $\mathbf{u}$ and $-\mathbf{v}$, given that it is in the range $(0°, 180°)$.

A) 30° B) 60° C) 120° D) 150° E) NOTA

11) Two sides of a parallelogram are given by the vectors $\langle 2, 4\rangle$ and $\langle 5, 1\rangle$. Find its area.

A) 9 B) 12 C) 14 D) 18 E) NOTA

12) If $A = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix}$, find the sum of the entries of $-A^{10}$.

A) 76 B) 78 C) 80 D) 82 E) NOTA

13) Find the sum of the three real eigenvalues of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{bmatrix}$.

A) -78 B) 0 C) 36 D) 100 E) NOTA

14) A sequence $\{M_n\}_{n\in\mathbb{N}_0}$ of matrices is given by $M_n = \begin{bmatrix} 2^{-n} & 3^{-n} \\ 4^{-n} & 5^{-n} \end{bmatrix}$. If the sum of the entries of $\sum_{n=0}^{\infty} M_n$ equals $\frac{A}{B}$, find $A + B$.

A) 37 B) 85 C) 137 D) 377 E) NOTA

15) The vectors $\langle 3, 2a-1, 4-b\rangle$, $\langle a+3b, -2, -4\rangle$ and $\langle 2a-b, -3, 0\rangle$ are pairwise perpendicular. Find $a + b$.

A) -1 B) 0 C) 1 D) 2 E) NOTA

16) For three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2022$. Find $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

A) -2022 B) $-\dfrac{1}{2022}$ C) $\dfrac{1}{2022}$ D) 2022 E) NOTA

17) A vector space is a set V with two operations ($+$ and $\cdot$) that must satisfy certain properties for all $a, b \in V$ and $c, d \in \mathbb{R}$. Which of the following is not one of those properties?

A) Additive Closure: $a + b \in V$

B) Additive Inverse: $\exists a' \in V$ such that $a + a' = 0_V$

C) Multiplicative commutativity: $a \cdot b = b \cdot a$

D) Multiplicative distributivity: $c \cdot (a + b) = c \cdot a + c \cdot b$

E) NOTA

18) Two lines are parametrized as follows, where $t \in \mathbb{R}$. Multiply the cosine of the smaller angle formed by their intersections by the value of t such that they intersect.

$$L_1 = \begin{cases} x = t + 4 \\ y = 2t - 3 \\ z = 5 - 2t \end{cases} \qquad L_2 = \begin{cases} x = 8t - 3 \\ y = 3 - 4t \\ z = t + 2 \end{cases}$$

A) $\dfrac{2}{27}$ B) $\dfrac{4}{27}$ C) $\dfrac{\sqrt{11}}{10}$ D) $\dfrac{\sqrt{11}}{5}$ E) NOTA

19) A sequence of points (a_1, b_1), (a_2, b_2), $(a_3, b_3)\ldots$ in the coordinate plane satisfies $(a_{n+1}, b_{n+1}) = \left(\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n\right)$ for all $n \in \mathbb{N}$. Suppose $(a_{100}, b_{100}) = (2, 4)$. Find $a_1 + b_1$.

A) $-\frac{3}{2^{98}}$ B) $-\frac{1}{2^{98}}$ C) $\frac{1}{2^{98}}$ D) $\frac{3}{2^{98}}$ E) NOTA

20) Find the smallest positive value of n such that $\begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $i = \sqrt{-1}$.

A) 6 B) 8 C) 12 D) 24 E) NOTA

21) Carlos the Croc is swimming across a 60-meter wide canal, heading to a point directly across from him (the line from Carlos to this point is perpendicular to the canal's current). Carlos can swim at 10 meters per second in still water, and the canal has a current of 5 meters per second. Find the minimum amount of time (in seconds) that it will take Carlos to cross the canal, given that he cannot walk on land.

A) $\frac{12}{\sqrt{5}}$ B) $4\sqrt{3}$ C) $\frac{18}{\sqrt{5}}$ D) $6\sqrt{3}$ E) NOTA

22) For three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} = 15$, $||\mathbf{a}|| = 4$, and $||\mathbf{b}|| = 5$. Find $||\mathbf{a} \times \mathbf{b}||$.

A) $5\sqrt{2}$ B) $5\sqrt{3}$ C) $5\sqrt{5}$ D) $5\sqrt{7}$ E) NOTA

23) The plane containing the points $(1, 1, 1)$, $(1, 7, 10)$, and $(5, 3, 1)$ also contains the point $(13, 12, z)$. Find z.

A) $\frac{15}{2}$ B) 8 C) $\frac{17}{2}$ D) 9 E) NOTA

24) Find the volume of a tetrahedron with vertices $(4, 3, 1)$, $(0, 4, 0)$, $(1, 0, 1)$, and $(4, 2, 0)$.

A) 3 B) 4 C) $\frac{9}{2}$ D) 6 E) NOTA

25) Find the sum of the entries of T, given the following.

$$T = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 9 & 7 & 9 & 3 & 2 & 3 \\ 3 & 3 & 8 & 3 & 2 & 7 \\ 4 & 1 & 9 & 7 & 1 & 6 \\ 5 & 1 & 0 & 5 & 8 & 2 \\ 6 & 7 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 6 & 5 & 3 & 5 & 9 \\ 8 & 4 & 6 & 2 & 6 & 4 \\ 9 & 5 & 0 & 2 & 8 & 6 \\ 9 & 3 & 9 & 9 & 3 & 7 \\ 0 & 9 & 7 & 4 & 9 & 1 \\ 7 & 8 & 1 & 6 & 4 & 4 \end{bmatrix}$$

A) 4500 B) 4600 C) 4700 D) 4800 E) NOTA

For questions $26-30$, you may use the following information.

$$M = \blacksquare$$

Error 2022.07.11: A printing error has occurred. 2-by-2 matrix of integers M not found. Answer choice "E) NOTA" not found, meaning it cannot be selected unless there is a dispute. Please refer to owner's manual to address issue.

Despite the issue, it remains the case that each of these questions has a unique answer.

26) M is which of the following?

A) Idempotent B) in Row-Echelon form

C) Singular D) Triangular

27) Which of the following statements is **not** true about M?

A) $M_{11} < M_{22}$ B) $M_{12} < M_{21}$

C) $M_{11} + M_{12} = M_{21} + M_{22}$ D) $M_{11} + M_{21} = M_{12} + M_{22}$

28) Find the sum of the entries of M.

A) 12 B) 16 C) 20 D) 30

29) Find $|M|$.

A) 12 B) 18 C) 24 D) 36

30) Find the trace of M.

A) 11 B) 13 C) 15 D) 19