Alpha Monumental Math Answers and Solutions

ANSWERS:

1) B

2) A

3) B

4) C

5) C

6) D

7) A

8) B

9) C

10) B

11) E (505)

12) C

13) D

14) A

15) D

16) A

17) C

18) D

19) D

20) B

21) A

22) B

23) D

24) A

25) E (no minimum)

26) C

27) C

28) A

29) B

30) C

SOLUTIONS:

1) Steve wants to visit 10 sites in 5 days. The number of sited he visits on each day is a random number between 1 and 3. We wish to find the number of ways five 3-sided dice can add to exactly 10. Through trial and error, there are three possibilities: 3+3+2+1+1=10 , 3+2+2+2+1=10, and 2+2+2+2+2=10. The order of each of these possibilities matters, so we need to count the number of distinct permutations of each possibility. The first has $\frac{5!}{2!2!} = 30$, the second has $\frac{5!}{3!} = 20$, and the final has 1 permutation, giving us 51 total permutations out of 3^5 possible dice throws. The probability is $\frac{51}{243} = \frac{17}{81}$ **B**

2) The Volume of the gem is $330\ carat \cdot \frac{200\ mg}{1\ carat} \cdot \frac{1\ g}{1000\ mg} \frac{1\ cm^3}{3.98g} = \frac{3300}{199}\ cm^3$. **A**

3) The number of ways we can choose at least two men is $\underbrace{\binom{7}{2}\binom{10}{2}}_{2\ men\ 2\ women} + \underbrace{\binom{7}{3}\binom{10}{1}}_{3\ men\ 1\ woman} + \underbrace{\binom{7}{4}\binom{10}{0}}_{4\ men\ 0\ women} = 1330$. There are $\binom{17}{4} = 2380$ possible subcommittees of size 4, therefore the probability is $\frac{1330}{2380} = \frac{19}{34}$ **B**

4) Set up the triangle and divide each side length by 100. We have via law of cosines $17^2 = 15^2 + 25^2 - 2(15)(25)\cos\theta$, which rearranges to $\cos\theta = \frac{187}{250}$ **C**

5) Writing this as a sine addition equation, we have $3\sin x + 4\cos x = 5\left(\frac{3}{5}\sin x + \frac{4}{5}\cos x\right) = 5(\sin x\cos\phi + \cos x\sin\phi) = 5\sin(x+\phi)$. The amplitude is 5 and the period is 2π, the product is then 10π. **C**

6) Consider SMITH to be one character, meaning there are 7 letters. The number of distinguishable permutations is them $\frac{7!}{2!2!2!}$ with the repetitions of S, I, and N. This number is then 630 **D**

7) $34°48' = \left(34\frac{48}{60}\right)^{\circ}$, which we then multiply by $\frac{\pi}{180}$ to get $34\frac{4}{5} \times \frac{\pi}{180} = \frac{29\pi}{150}$ **A**

8) The radius of the sphere is equal to the space diagonal of the rectangular prism. We can compute this by extending the Pythagorean theorem to $d^2 = a^2 + b^2 + c^2$ where $a = \frac{13}{4}, b = \frac{13}{4}, c = \frac{5}{4}$ (half of each side length given in the problem). This leads to $d = \sqrt{\left(\frac{13}{4}\right)^2 + \left(\frac{13}{4}\right)^2 + \left(\frac{5}{4}\right)^2} = \frac{11\sqrt{3}}{4}$. The volume of the sphere is then $V = \frac{4}{3}\left(\frac{11\sqrt{3}}{4}\right)^3\pi = \frac{1331\sqrt{3}}{16}\pi$ **B**

9) See the diagram below. To solve for x, we use the property of 30-60-90 triangles that the longer leg is $\sqrt{3}$ times longer than the shorter leg, so $x = 19\sqrt{3}$. To find the length of the entire base $(x+y)$, we use the fact that $\tan(15°) = 2 - \sqrt{3} = \frac{19}{y+19\sqrt{3}}$. We can rearrange and solve for y as follows: $(2-\sqrt{3})(y+19\sqrt{3}) = 19 \rightarrow 19 = 2y - y\sqrt{3} + 38\sqrt{3} - 57 \rightarrow y = \frac{76-38\sqrt{3}}{2-\sqrt{3}} = 38$ **C**

10) $r_1 = 36$ and $r_2 = 8$, because there are two dividers with 4 each. Then the quadratic has roots $\sqrt{r_1} = 6$ and $r_2^{2/3} = 4$. It can be written as $f(x) = k(x-4)(x-6)$. Plug in $x = 7$ and $y = -81$ to get $-81 = k(7-4)(7-6) \rightarrow 3k = -81 \rightarrow k = -27$ **B**

11) The solutions to $z^n - k = 0$ will lie evenly spaced around a circle with radius $\sqrt[n]{k}$ centered at the origin, and if $k \in \mathbb{R}$ one of the roots will lie along the positive $x-$axis. This means that the 2022 roots in the equation, one will be on the positive and negative $x-$axis, and the remaining 2020 will be evenly split between the 4 quadrants. $2020 \div 4 = 505$. **E**

12) The absolute value here just ensures each hump is above the x-axis. For each hump, since the graph is being multiplied by 2, the area underneath is doubled so each hump is now area 4. We subtract this from the area of a box with dimensions $\underbrace{4}_{height} \times \underbrace{\pi}_{width}$, for an area of $4\pi - 4$ or $4(\pi - 1)$. There are 2022 of these units we must consider, therefore the total area is $2022 \cdot 4(\pi - 1) = 8088(\pi - 1)$ **C**

13) First, note that $\cos(n\pi) = \pm 1$ alternating. Writing out a few terms, we have $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$, then the period of $\sin\left(\frac{2}{3}x\right)$ is $\frac{2\pi}{2/3} = 3\pi$ **D**

14) Let $q = \frac{18^{56}}{56^{18}}$. If we can show q is larger than 1, then we have that k is larger. Write each base as the product of its prime factors, then use exponent rules to simplify

$$\frac{(2 \cdot 3^2)^{56}}{(2^3 \cdot 7)^{18}} = \frac{2^{56} \cdot 3^{112}}{2^{54} \cdot 7^{18}} = \frac{2^2 \cdot 3^{112}}{7^{18}} > \frac{2^2 \cdot 3^{112}}{6^{18}} = \frac{2^2 \cdot 3^{112}}{2^{18} \cdot 3^{18}} = \frac{3^{94}}{2^{18}}$$

Which is clearly larger than 1. So k is larger **A**

15) L is the difference between the max and min of the function, which is just twice the amplitude so $l = 36$. The minimum is the height of the Midline minus the amplitude of the function so $X = 12 - 18 = -6$. Then we have $Le^X = 36e^{-6}$ **D**

16) By Law of Cosines, $x^2 = 247^2 + 247^2 - 2(247)(247)(-0.57) = 247^2 \cdot 2(1 + 0.57) = 247^2(3.14)$. Take the square root of both sides to get option **A**

17) $x = (58320)^{1/8}$ so let us examine some 8th powers. $1^8 = 1, 2^8 = 256, 3^8 = 6561, 4^8 = 65536$. We see that 58,320 is pretty close to 4^8 so $(58320)^{1/8} = x$ should be pretty close to $(4^8)^{1/8} = 4$, putting it on the interval shown in **C**

18) The area of the circle that is not in the triangle is $\frac{5}{6}\pi r^2$ because sector covered by the triangle has central angle $\pi/3$, equal to$5/6$ of the total area of the circle. The area of the shape is that area we just wrote down plus the area of the triangle (side length $3r$), which is $\frac{(3r)^2\sqrt{3}}{4} + \frac{5}{6}\pi r^2$ The probability is then

$$\frac{\frac{5}{6}\pi r^2}{\frac{5}{6}\pi r^2 + \frac{9r^2\sqrt{3}}{4}} = \frac{10\pi}{10\pi + 27\sqrt{3}}$$

Which is option **D**

19) Option **D**

$$\sin(2^{2023}x) = \cos x \cos 2x \cos 4x \cdots \cos 2^{2022}x$$

$$\sin x \sin(2^{2023}x) = \sin x \cos x \cos 2x \cos 4x \cdots \cos 2^{2022}x$$

$$\sin x \sin(2^{2023}x) = \frac{1}{2}\sin 2x \cos 2x \cos 4x \cdots \cos 2^{2022}x$$

$$\sin x \sin(2^{2023}x) = \frac{1}{4}\sin 4x \cos 4x \cdots \cos 2^{2022}x$$

$$\vdots$$

$$\sin x \sin(2^{2023}x) = \frac{1}{2^{2023}}\sin(2^{2023}x)$$

$$\sin x = \frac{1}{2^{2023}} \to x = \sin^{-1}\left(\frac{1}{2^{2023}}\right)$$

20) We know the major radius is $a = 190$, and the focal radius is $c = 15\sqrt{91} = \sqrt{a^2 - b^2}$. We need to plug in a into the focal radius equation to solve for b like so

$$c^2 = 20475 = 190^2 - b^2$$

$$b^2 = 190^2 - 20475$$

$$b^2 = 36100 - 20475 = 15625 = 125^2$$

$$b = 125$$

The area is then $ab\pi = 190 \cdot 125\pi = 23750\pi$ Option **B**

21) Call the midpoint of $\overline{C_{NE}C_{NW}}$ K. It is apparent that K lies on the line segment $\overline{A_N J}$, so we'll reduce the given info to $m\angle KJC_{NW} = \theta$. Because K is the midpoint, then $KC_{NW} = \frac{1}{2}C_{NE}C_{NW} = 110$. If we draw out the triangle given by $\tan\theta = \frac{10}{9}$, the side length that corresponds to $\overline{KC_{NW}}$ has length 10, so we just need to scale that up by a factor of 11 to match the real monument, giving us side lengths of 99 and 110, the sum of which is 209. This sum is one quarter of the overall perimeter, so $4 \cdot 209 = 836$, option **A**

22) The given equation is in the form $r = \frac{a(1-e^2)}{1 \pm e\sin\theta}$, and in this form the length of the latus rectum is twice the numerator, $2a(1-e^2) \to 2(100)\left(1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = 50$, option **B**

23) $y = \tan 18x$ has 18 roots on the interval $[0, \pi)$, and $y = \tan 14x$ has 14 roots on the same interval. We need to add these two numbers together to geth the number of roots of $y = \tan 18x \tan 14x$, but we accidentally counted the root at $x = 0$ and the one at $x = \frac{\pi}{2}$ twice! So we have $18 + 14 - 2 = 30$ roots on the interval $[0, \pi)$. The total interval is 1812π long so we just multiply $30 \cdot 1812 = 54{,}360$ roots on the total interval, $5 + 4 + 3 + 6 = 18$, option **D**

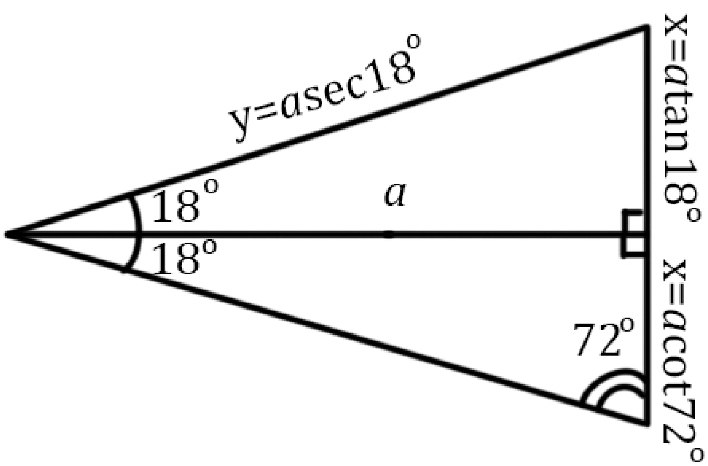
24) In a limit where the variable gets infinitely large, $\sqrt{n+8n}$ is essentially the same as $\sqrt{n^2+8n+16}=n+4$, and the same can be said for $\sqrt{n^2-14n}$ being essentially $\sqrt{n^2-14n+49}=n-7$. Our limit becomes

$$\frac{L}{7}=\lim_{n\to\infty}\sqrt{n^2+8n}-\sqrt{n^2-14n}=n+4-(n-7)=11\Rightarrow L=77\Rightarrow 10L+7=777$$

Which is Option **A**

25) The eccentricity of a hyperbola is $e>1$, however there is no "smallest number greater than 1", therefore there is no minimum of the eccentricity of a hyperbola. It is bounded below, but that doesn't imply a minimum. **E**

26) Please refer to the diagram below, showing one tenth of a decagon. Note: $18°=\frac{\pi}{10}$ rad and $72°=\frac{2\pi}{5}$ rad



The first way we can compute the area is the base a and the height of the top triangle $x=a\tan\frac{\pi}{10}$. Area would be ½ of their product, multiplied by 20, giving us option A. Option B is the same thing but we swap the base in terms of tan for the base in terms of cot. Option D is if we use the $\frac{1}{2}ab\sin C$ formula with both sides equal to $a\sec\frac{\pi}{10}$ and there being 10 of these triangles. Option **C** is the only one that does not give the area of the decagon

27) The left-hand side of the equation is a geometric series with first term $1-\sin x$ and common ration $1-\sin x$. The sum is then equal to $\frac{1-\sin x}{1-(1-\sin x)}=\frac{1-\sin x}{\sin x}$. We can sub $y=\sin x$ for simplicity and we have $\frac{1-y}{y}=2y\rightarrow 2y^2+y-1=0$ $\rightarrow(2y-1)(y+1)=0$ so we either have $\sin x=\frac{1}{2}$ or $\sin x=-1$. The latter cannot be true as when we plug back into the original we get a divergent series, so we have $x=30°,150°$ for a sum of $180°$ so $j=18$, and $\frac{2j}{3}=12$ option **C**

28) We must find k first. The first modular equation gives $(*)\ \ k=7j+1$ for some integer j. plugging in to the second mod gives $7j+1\equiv 2 \bmod 9\Rightarrow 7j\equiv 1 \bmod 9$. If we multiply both sides by 4 we get $28j\equiv 1j\equiv 4 \bmod 9$ which leads us to $j=9l+4$ for some integer l. We can then plug this in to $(*)$ to get $k=63l+29$. Plug this new expression of k into the last mod to get $63l+29\equiv 4 \bmod 13\Rightarrow 63l\equiv 1 \bmod 13\Rightarrow 11l\equiv 1 \bmod 13$. Here if you multiply both sides by 6, you get $66l\equiv 1l\equiv 6 \bmod 13$, meaning $l=13t+6$. Finally plug this back in to k, to get $k=819t+407$, the smallest value of this is when $t=0$ so we take $16k=16\cdot 407=6512$ **A**

29) Assign each vertex the number of different paths one can take before having to leave the vertex. Then the grid looks as follows:

16	128	640	2560
8	48	192	640
4	16	48	128
2	4	8	16

The first spot has two options because Kim can either stay or leave. Each of the spots she can reach have two options as well, stay for the sights or leave immediately. This is equivalent to doing the standard Pascal's triangle additions to find the number of paths, then multiply by 2. Filling in the grid upwards gives us $n = 2560 = 2^9 5^1 \Rightarrow \sqrt{9} + \sqrt{1} = 3 + 1 = 4$ **B**

30) We simply take $\frac{6!}{2!} = 360$ **C**