

For this test, E) NOTA means “None Of These Answers”. All inverse trigonometric functions will be restricted to their traditional domains and ranges. The constant $i = \sqrt{-1}$, and unless otherwise stated, logarithms will only accept positive inputs and output a real number. Good luck!

Welcome to Washington DC! This tour guide will give you lots of important historical information about the sites, sounds, and tastes of America’s capitol city. Each location highlighted on our tour of the District will have a selection of math problems, brain teasers, or riddles associated with the story of the history of this great city. We hope you enjoy DC!

We will begin our journey with the portal to American History, the Smithsonian Museum!

1) The Smithsonian has 19 different museums, 21 libraries, 9 research centers and a zoo. Steve wants to visit all the research centers and the zoo. He visits a random number of these sites per day, at least 1 but at most 3. Assuming the number of sites he visits daily is uniformly distributed (each has equal probability), and he does not repeat any sites, what is the probability that Steve visits all of the sites he wants to visit in exactly 5 days?

- A) $\frac{41}{243}$ B) $\frac{17}{81}$ C) $\frac{61}{243}$ D) $\frac{71}{243}$ E) NOTA

2) The Museum of Natural History has within it the Hall of Geology, Gems, and Minerals. It houses some of the most important and valuable gems and samples of minerals in the country. One such gem is the Star of Asia, a roughly spherical sapphire weighing in at 330 carat! Given that 1 carat = 200 mg, and the density of sapphire is $3.98 \frac{g}{cm^3}$, what is the volume of the Star of Asia, in cubic centimeters?

- A) $\frac{3300}{199} cm^3$ B) $\frac{3300}{99} cm^3$ C) $66 cm^3$ D) $\frac{199}{50} cm^3$ E) NOTA

3) The advisory board of the Smithsonian Libraries consists of 10 women and 7 men. They wish to convene a subcommittee consisting of 4 members to discuss the opening of a 22nd library site. What is the probability that this committee contains at least 2 men?

- A) $\frac{3}{17}$ B) $\frac{19}{34}$ C) $\frac{19}{51}$ D) $\frac{14}{17}$ E) NOTA

4) The distance from the Washington Monument (point W) to the Museum of American History (point A) is 1500 ft. The distance from the Washington Monument to the Smithsonian Castle (point C) is 2500 ft. Given that the distance from A to C is 1700 ft, find the measure of $\cos(\angle AWC)$.

- A) $\frac{1}{4}$ B) $\frac{581}{2250}$ C) $\frac{187}{250}$ D) $\frac{601}{2250}$ E) NOTA

5) A certain display in the Museum of American History is dedicated to the advent of the radio, and the effect it had on the home lives of millions of Americans. In the art surrounding the display, a sinusoidal wave is shown, and determined to have equation $y = 3 \sin x + 4 \cos x$. What is the product of the amplitude and the period of this function?

- A) 2π B) 5π C) 10π D) 24π E) NOTA

6) How many distinguishable ways are there to arrange the letters of SMITHSONIAN if the letters SMITH must appear somewhere in the word in that order?

- A) $\frac{11!}{2!2!2!}$ B) 5040 C) 720 D) 630 E) NOTA

Our next stop on our tour of DC is the Washington Monument and the Lincoln Memorial!

7) The Washington monument towers over the national mall, coming in at 554 feet, $7\frac{11}{32}$ inches. It's a hollow obelisk, where horizontal cross sections appear to be square, and it has a hollow pyramidion (a hollow pyramid with no bottom, frequently a capstone on Egyptian pyramids) on top. A vertical cross section of the pyramidion is taken such that the plane of the slice passes through the apex and is perpendicular to opposite sides of the pyramidion. The angle that two opposite sides form on this cross section is $34^\circ 48'$. Convert this quantity to radians.

- A) $\frac{29\pi}{150}$ B) $\frac{31\pi}{150}$ C) $\frac{150\pi}{29}$ D) $\frac{150\pi}{31}$ E) NOTA

8) The cornerstone (first piece placed above ground) of the Washington monument is a massive 24,500 pound marble block that measures 2.5 feet by 6.5 feet by 6.5 feet. Imagine a sphere is circumscribed around such a stone, find the volume of this sphere in cubic feet.

- A) $\frac{121\sqrt{3}}{8}\pi$ B) $\frac{1331\sqrt{3}}{16}\pi$ C) $\frac{11\sqrt{3}}{4}\pi$ D) $\frac{\sqrt{33}}{2}\pi$ E) NOTA

9) The Lincoln Memorial contains within it a statue of Abraham Lincoln, the 16th president of the United States. The statue is 19 feet tall. Kiley is standing in a position such that the angle of elevation from her feet to the top of the Lincoln statue is 30° . If she moves back to a position where the angle of elevation between her feet and the top of the statue is now 15° , how far did she move back in feet?

- A) 19 B) $19\sqrt{3}$ C) 38 D) $38\sqrt{3}$ E) NOTA

10) There are 36 Doric columns on the outside of the building, one for each state that was in the Union at the time of Lincoln's death. The inner part of the building is separated into three chambers by two rows of 4 Ionic columns. Let r_1 = the number of Doric columns and r_2 = the number of Ionic columns. The quadratic function $f(x)$ has roots $\sqrt{r_1}$ and $(r_2)^{2/3}$. What is the leading coefficient of $f(x)$ if $f(7) = -81$?

- A) -9 B) -27 C) -81 D) -243 E) NOTA

11) There's all sorts of rumors and urban legends about the statue of Abraham Lincoln, such as a theory that General Robert E Lee's face is carved on the back of Lincoln's head. Another (that I definitely didn't invent for the purpose of this problem) is that there's a Complex Plane inscribed on Lincoln's ankle, and on it are graphed all of the complex solutions to $z^{2022} - 2022 = 0$. How many of these solutions lie within the 3rd quadrant?

- A) 2022 B) 2021 C) 2020 D) 2019 E) NOTA

12) A question not related to a monument! Given that the area underneath one 'hump' of the sine graph is 2 square units, how much area is enclosed by the shape defined by the line $x = 0$, the line $y = 4$, the line $x = 2022\pi$, and the graph of $y = 2|\sin x|$?

- A) $8088(\pi + 1)$ B) 8088π C) $8088(\pi - 1)$ D) $8088(\pi - 2)$ E) NOTA

13) Another non-monumental question! What is the period of the function $f(x + 2) = \sin(kx)$, where

$$k = \sum_{n=0}^{\infty} \frac{\cos(n\pi)}{2^n}$$

- A) $\frac{2}{3}$ B) $\frac{2}{3}\pi$ C) 3 D) 3π E) NOTA

14) The Old Ebbitt Grill is Washington's oldest saloon, having been founded in 1856. Which of the following is larger? $k = 18^{56}$ or $t = 56^{18}$

- A) k is larger B) t is larger C) $k = t$ D) Do not select D E) NOTA

15) The number of Presidential Bobbleheads that the White House Gift Shop sells in a day is modeled by the function $P = 18[\sin(\pi t/2)] + 12$. The difference between the maximum and minimum number of bobbleheads sold on any day in a given week is L , and the minimum number sold is X . Find Le^X . For this question, $\lfloor x \rfloor$ means round x down to the nearest integer. Assume $t \in \mathbb{N}$.

- A) $9e^{-3}$ B) $16e^{-4}$ C) $25e^{-5}$ D) $36e^{-6}$ E) NOTA

Now we will take a trip to the War Memorials on the National Mall.

16) The Vietnam Veterans Memorial is a large, black granite wall that is engraved with the names of those who died as a result of the Vietnam War. The wall is split in half, and the two halves form a 125° angle, and each half is 247 feet long. Which of the following is the distance between the two endpoints of the wall, given that $\cos(125^\circ) \approx -0.57$?

- A) $247\sqrt{3.14}$ B) $247\sqrt{2}$ C) $140.79\sqrt{1.57}$ D) $3.14\sqrt{247}$ E) NOTA

17) The Vietnam Veterans Memorial currently contains the names of 58,320 service members, including 8 women.

On what interval does $x = \sqrt{\sqrt{\sqrt{58,320}}}$ lie?

- A) $1.5 \leq x < 2.5$ B) $2.5 \leq x < 3.5$ C) $3.5 \leq x < 4.5$ D) $4.5 \leq x < 5.5$ E) NOTA

18) The Korean War Veterans Memorial is in the shape of a triangle that intersects a circle. The triangle is equilateral and the center of the circle is one of the vertices of the triangle. The radius of the circle is r and the side length of the triangle is $3r$. What is the probability that a randomly selected point on the interior of this shape is within the circle but not the triangle?

- A) $\frac{12\pi}{27\sqrt{3}+12\pi}$ B) $\frac{12\pi}{27\sqrt{3}+10\pi}$ C) $\frac{10\pi}{27\sqrt{3}+12\pi}$ D) $\frac{10\pi}{27\sqrt{3}+10\pi}$ E) NOTA

19) While reading the directory book listing the images on the wall, Kyle notices an equation scribbled down hastily. It reads $\sin(2^{2023} \cdot x) = \prod_{k=0}^{2022} \cos(2^k \cdot x)$. Which of the following values of x is a solution to the equation above?

- A) $\frac{1}{2^{2022}}$ B) $\sin\left(\frac{1}{2^{2022}}\right)$ C) $\frac{1}{2^{2023}}$ D) $\sin^{-1}\left(\frac{1}{2^{2023}}\right)$ E) NOTA

20) The World War II memorial is laid out in the shape of an ellipse, with major axis (running north-south) 380 ft. long. Given that the focal radius is $15\sqrt{91}$, find the area of the World War II memorial in square feet.

- A) $20,475\pi$ B) 23750π C) $36,100\pi$ D) Multiple Answers E) NOTA

21) The World War II memorial consists of 56 columns, each representing a US State or Territory from 1945, and two arches, one at the northernmost point and one at the southernmost. Each quadrant of the ellipse has a group of 14 columns, beginning at the arch and spreading towards the east-west line through the center. Within each group, the columns are equally spaced, but the arcs of columns don't quite reach the east-west centerline. Call the last column in each group $C_{NE}, C_{NW}, C_{SE}, C_{SW}$, depending on which ordinal quadrant they lie in, the arches A_N, A_S , and the center of the ellipse J . Given $m\angle A_N J C_{NW} = \theta$ where $\tan \theta = \frac{10}{9}$, and $C_{NE} C_{NW} = 220$ ft, find the perimeter of the quadrilateral $C_{NE} C_{NW} C_{SW} C_{SE}$.

- A) 836 ft B) 880 ft C) 418 ft D) 440 ft E) NOTA

22) There is a smaller ellipse inside the World War II memorial, the outline of the fountain in the center of the monument. It has a major axis that has length 200 ft, and the ellipse has equation $r = \frac{100\left(1 - \left(\frac{\sqrt{3}}{2}\right)^2\right)}{1 + \frac{\sqrt{3}}{2} \sin \theta}$. What is the length of the latus rectum of this ellipse?

- A) 25 ft B) 50 ft C) 75 ft D) 100 ft E) NOTA

For questions 23 – 26, we're making a trip to the White House, the residence of the Presidents!

23) The original White House was burned to the ground in 1814 by the British troops during the War of 1812. If the function $f(x) = \tan(18x) \cdot \tan(14x)$ has q roots on the interval $[0, 1812\pi)$, find the sum of the digits of q .

- A) 33 B) 32 C) 19 D) 18 E) NOTA

24) The amount of windows on the Executive Residence (middle building) of the White House is equal to L . It is known that $\frac{L}{7} = \lim_{n \rightarrow \infty} \sqrt{n^2 + 8n} - \sqrt{n^2 - 14n}$. What is $10L + 7$?

- A) 777 B) 4557 C) 1547 D) 2187 E) NOTA

25) Suppose time and space are being stretched by the evil Rob Snow. He transforms space-time so that the Oval Office is now the Hyperbola Office! What is the minimum value of the eccentricity of the new Hyperbola Office?

- A) 2 B) $\sqrt{2}$ C) 1 D) 0 E) NOTA

26) The white house is, of course, protected by the secret service. The badge of the secret service contains on it a decagon. Which of the following does NOT represent the area of a regular decagon with apothem a ?

- A) $10a^2 \tan\left(\frac{\pi}{10}\right)$ B) $10a^2 \cot\left(\frac{2\pi}{5}\right)$ C) $5a^2 \sin\left(\frac{\pi}{5}\right)$ D) $5a^2 \sec^2\left(\frac{\pi}{10}\right) \sin\left(\frac{\pi}{5}\right)$ E) NOTA

Here are two questions about the International Spy Musuem!

27) The International Spy Museum (ISM) houses over 7,000 artifacts pertaining to espionage, and one particular exhibit showcases the many different cell phones that one particular agent had to juggle incoming and outgoing information. Given the secretive nature, the messages would be sent and received in ciphers. The number of phones on display is j , and for x measured in degrees, $0^\circ < x < 360^\circ$, we know that the sum of the solutions to $\sum_{k=1}^{\infty} (1 - \sin(x))^k = 2\sin(x)$ is equal to $10j$. Give the value of $\frac{2j}{3}$.

- A) 2 B) 18 C) 12 D) 30 E) NOTA

28) The number of spies and covert operatives that various countries have deployed at any given time is a mystery. When world powers cease to exist, some of their documents detailing this number are released, and of course they're as cryptic as can be. The number of spies that the former Soviet Union employed in 1990 was $16k$, where k is the smallest positive integer that satisfies $k \equiv 1 \pmod{7}$, $k \equiv 2 \pmod{9}$, and $k \equiv 4 \pmod{13}$. How many spies did they employ?

- A) 6512 B) 13,104 C) 407 D) 10,269 E) NOTA

We end our tour of Washington DC at the Capitol Building.

29) The Capitol sits along the eastern end of the National Mall, and was once the geographic center of Washington D.C. The District has expanded, but the Capitol building does still form the Origin point for Washington D.C.'s street numbering system and the district's 4 quarters. Suppose the street system near the Capitol is a grid, with street numbers increasing as the distance to the capital increases (1st runs right into the Capitol, 2nd street is one block further, and so on). Suppose Kim is walking from the intersection of 4th St and 4th Ave in the South West quadrant, and at each intersection (including her starting position), she can perform one of three actions: She can move North one block, she can move East one block, or she can stay at the current intersection to see the sights (she can only perform this last action once per intersection). The number of different paths she can take to get to the Capitol building is $n = 2^a 5^b$, compute $\sqrt{a} + \sqrt{b}$.

- A) 7 B) 4 C) 6 D) 5 E) NOTA

30) Congratulations on making it to the end of the test! Your reward is an easy problem about the Capitol! Washington DC is surrounded by Virginia and Maryland, and the metropolitan area is called the DMV. How many distinguishable permutations exist for the slightly extended initialism DCMDVA?

- A) 180 B) 270 C) 360 D) 720 E) NOTA