1. D	7. A	13. B	19. C	25. D
2. D	8. C	14. D	20. E	26. B
3. C	9. B	15. D	21. E	27. B
4. B	10. A	16. C	22. D	28. C
5. B	11. B	17. A	23. B	29. A
6. C	12. A	18. A	24. A	30. B

1. Let θ be an angle satisfying

$$\sin^6(\theta) + \cos^6(\theta) = \frac{2021}{2022}$$

What is the value of $\sin^2(\theta) + \cos^2(\theta)$?

A. $\frac{2020}{6063}$ B. $\frac{1}{3}$ C. $\frac{2020}{2021}$ $\boxed{\text{D. } 1}$ E. NOTA

Solution: For any angle that could possibly satisfy the initial equation $\sin^2(\theta) + \cos^2(\theta) = 1$

---

2. The espression $\sin(\theta) + 2\cos(\theta)$ can also be written in the form $-\sqrt{5}\sin(\theta + \phi)$, for some $\phi \in [0, 2\pi)$. What is the value of $\tan(\phi)$?

A. -2 B. $-\frac{1}{2}$ C. $\frac{1}{2}$ $\boxed{\text{D. } 2}$ E. NOTA

Solution: Using sine angle addition, the quantity of interest can be written as $-\sqrt{5}\sin(\theta)\cos(\phi) - \sqrt{5}\cos(\theta)\sin(\phi)$. For the two expressions to be equivalent, we need $\sin(\phi) = -\frac{2}{\sqrt{5}}$ and $\cos(\phi) = -\frac{1}{\sqrt{5}}$. From these values, we get $\tan(\phi) = 2$.

---

3. Which of the following quadratic functions have roots $\cos(\theta)$ and $\sin(\theta)$ for some angle $\theta \in [0, 2\pi)$?

A. $f(x) = x^2 - x - 1$ B. $f(x) = x^2 + x + 1$
$\boxed{\text{C. } f(x) = 8x^2 - 4x - 3}$ D. $f(x) = 8x^2 - 4x + 3$ E. NOTA

Solution: There are many ways to approach this problem, but one shortcut is to note that if $\cos(\theta)$ and $\sin(\theta)$ are roots, then

$$f(\cos(\theta)) = 0, f(\sin(\theta)) = 0$$

Therefore the sum of these quantities must be zero as well. By doing this for each possible banswer, only A and C are possible because of the domain of $\cos(\theta)$ and $\sin(\theta)$, but we can quickly rule out A because we note that $-1, 0$ are not the roots of the polynomial. We can check C using whichever method we prefer to see that it indeed can have roots that satisfy this condition.

---

4. The point $(4, 2)$ is rotated $\frac{\pi}{3}$ radians clockwise around the origin, then reflected over the $y-$axis. The result of these transformations is the point (a, b). Compute the value of $|ab|$.

A. $3\sqrt{3} - 4$ $\boxed{\text{B. } 3\sqrt{3} + 4}$ C. $6\sqrt{3}$ D. $6\sqrt{3} + 3$ E. NOTA

Solution: We know that the Cartesian point $(4, 2)$ corresponds to the complex number $4 + 2i$ when plotted in the complex plane. A rotation of $\frac{\pi}{3}$ clockwise can be done by multiplying by the complex number $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and a reflection across the $y-$axis can be done by inverting the sign of the real compononet. the result of these two operations is $(-2 - \sqrt{3}) + (1 - 2\sqrt{3})i$, and the product of the real and imaginary parts is $3\sqrt{3} + 4$

---

5. The curve

$$y = x + \frac{\sin(2x)}{4}$$

along with the line $x = \pi$ and the $x-$axis bound a region in the $xy-$plane. What is the area of this region?

A. $\frac{\pi^2}{4} + \frac{\pi}{2}$ B. $\boxed{\frac{\pi^2}{2}}$ C. $\frac{\pi^2}{2} + \frac{\pi}{8}$ D. $\frac{\pi^2}{4} + \pi$ E. NOTA

Solution: Note that from 0 to π, $\sin(2x)$ completes exactly one period. If we were to draw the line $y = x$ alongside the curve in question, we would note that the area of the curve above the line is the same as the area of the curve below the line. Thus the area of the entire figure (if we "fill in holes") is just the area under the line, which is $\frac{\pi^2}{2}$

---

6. For how many angles $\theta \in [0, 2\pi)$ is the following matrix singular?

$$\begin{bmatrix} \sin(\theta) & 3\cos(\theta) \\ \cos(\theta) & 2\sin(\theta) \end{bmatrix}$$

A. 0 B. 2 $\boxed{\text{C. } 4}$ D. 8 E. NOTA

Solution: For this matrix to be singular, it must have a zero determinant. If we compute the determinant of this matrix, we see that it is equal to

$$2\sin^2(\theta) - 3\cos^2(\theta) = 2 - 5\cos^2(\theta)$$

after applying the Pythagorean Identity. For this wuantity to equal 0, $\cos(\theta) = \pm\frac{2}{5}$, which happens 4 times on the interval

---

7. What is the area enclosed by the graph of $r = \sin(\theta) + 3\cos(\theta)$ in the polar plane?

$\boxed{\text{A. } \frac{5\pi}{2}}$ B. 3π C. $\frac{9\pi}{2}$ D. 5π E. NOTA

Solution: Note that if we multiply both sides by r, we can easily convert to rectangular and find that

$$x^2 + y^2 = y + 3x \Rightarrow \left(x - \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{2}$$

So this is a circle with area $\frac{5\pi}{2}$

---

8. Quadrilateral $ABCD$ satisfies $AD = 3$, $BC = 4$, $AB = 6$, with $m\angle DAB = m\angle ABC = \frac{\pi}{3}$. If DC has length x, compute x^2.

   A. 5 B. 6 $\boxed{\text{C. } 7}$ D. 8 E. NOTA

   Solution: Extend AD and BC to meet at a constructed point E. Because both base angles are $\frac{\pi}{3}$, angle AEB must also measure $\frac{\pi}{3}$ and triangle AEB must be equilateral with side length 6. Therefore ED = 3 and EC = 2, and by Law of Cosines,

$$DC^2 = 3^2 + 2^2 - 12 \cdot \frac{1}{2} \Rightarrow DC^2 = 7$$

---

9. Let $\theta \in [0, 2\pi)$ satisfy

$$\sin^4(\theta) + \cos^4(\theta) = \frac{2}{3}$$

   Compute the value of $\sin^8(\theta) + \cos^8(\theta)$.

   A. $\frac{1}{3}$ $\boxed{\text{B. } \frac{7}{18}}$ C. $\frac{4}{9}$ D. $\frac{1}{2}$ E. NOTA

   Solution: Note thet $\sin^2(\theta) + \cos^2(\theta) = 1$, so we have

$$(\sin^2(\theta) + \cos^2(\theta))^2 = \sin^4(\theta) + 2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta) = 1 \Rightarrow 2\sin^2(\theta)\cos^2(\theta) = \frac{1}{3}(*)$$

   We can do a similar thing on the fourth power equation to get

$$2\sin^4(\theta)\cos^4(\theta) + Q = \frac{4}{9}$$

   where Q is the desired quantity. We can square (*) and divide by 2 to get that $Q = \frac{4}{9} - \frac{1}{18} = \frac{7}{18}$

---

10. For a real number x, compute the minimum possible value of the expression

$$\sin(x) - 2\cos^2(x)$$

   $\boxed{\text{A. } -\frac{17}{8}}$ B. -2 C. $-\frac{15}{8}$ D. $-\frac{3}{2}$ E. NOTA

   Solution: We can rewrite the equation using the Pythagorean Theorem to see our expression is equivalent to $2\sin^2(x) + \sin(x) - 2$. This is a quadratic function in $\sin(x)$ that is minimized at $\sin(x) = -\frac{1}{4}$. Plugging in this value gives us a minimum of $-\frac{17}{8}$.

---

11. For how many real angles $\theta \in [0, 2\pi)$ is the following equation satisfied?

$$\sum_{n=0}^{\infty} \sin(\theta)(-\cos(\theta))^n = \sqrt{2}$$

A. 0 $\boxed{\text{B. } 1}$ C. 2 D. 3 E. NOTA

Solution: Using the sum for an infinite geometric series (and noting that $\theta \neq 0$ as $\cos(\theta)$ must be strictly less than 1) we have

$$\frac{\sin(\theta)}{1+\cos(\theta)} = \sqrt{2}$$

Recognizing the LHS as $\tan(\frac{\theta}{2})$, we see that there is only 1 value that works between $\frac{\pi}{2}$ and π.

---

12. An isosceles triangle ABC has vertex at B with $m\angle B = \phi$ and $\sin(\phi) = \frac{2}{5}$. If $AC = 4$, what is the area of the circumcircle of ABC?

$\boxed{\text{A. } 25\pi}$ B. 32π C. 40π D. 50π E. NOTA

Solution: Using the Extended Law of Sines, we see that

$$2R = \frac{4}{\frac{2}{5}} \Rightarrow R = 5$$

therefore the circle has area 25π

---

13. Jeffrey's triangle JLU has the interesting property that its area satisfies

$$A(\theta) = \sin(2\theta)$$

where $A(\theta)$ is the area function which depends on $\theta = m\angle JLU$. If $JL = \sqrt{2}$ and $LU = \sqrt{3}$, then what is the length of UJ?

A. 1 $\boxed{\text{B. } \sqrt{2}}$ C. $\sqrt{3}$ D. 2 E. NOTA

Solution: Note that the area of Jeffrey's triangle is also

$$\frac{1}{2}\sqrt{2}\sqrt{3}\sin(\theta)$$

Letting the two quantities be equal and aplying the sine double angle formula we have

$$2\sin(\theta)\cos(\theta) = \frac{\sqrt{6}}{2}\sin(\theta) \Rightarrow \cos(\theta) = \frac{\sqrt{6}}{4}$$

Now using the Law of Cosines, we have

$$UJ^2 = 2 + 3 - 2\cdot\sqrt{2}\cdot\sqrt{3}\cdot\frac{\sqrt{6}}{4} \Rightarrow UJ = \sqrt{2}$$

---

14. What is the distance between the points $\left(2, \frac{2\pi}{3}\right)$ and $\left(4, \frac{\pi}{6}\right)$, which are both in polar coordinates?

A. $\sqrt{10}$ B. $2\sqrt{3}$ C. $3\sqrt{2}$ $\boxed{\text{D. } 2\sqrt{5}}$ E. NOTA

Solution: Noting that the angle difference is $\frac{\pi}{2}$, we can use the Pythagorean Theorem to see that the distance between the points is $2\sqrt{5}$

---

15. If $\tan(x) = \frac{1}{2}$, what is the sum of all possible distinct values of $\tan(3x)$?

A. 0 B. $\frac{7}{2}$ C. $\frac{9}{2}$ $\boxed{\text{D. } \frac{11}{2}}$ E. NOTA

Solution: First compute

$$\tan(2x) = \frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}} = \frac{4}{3}$$

Now, use angle addition to get

$$\tan(3x) = \tan(2x + x) = \frac{\frac{1}{2} + \frac{4}{3}}{1 - \frac{1}{2} \cdot \frac{4}{3}} = \frac{11}{2}$$

---

16. Given that

$$\arcsin(x)\arccos(x) = \frac{1}{6}$$

compute the value of

$$\frac{1}{\arcsin(x)} + \frac{1}{\arccos(x)}$$

A. 3 B. 6 $\boxed{\text{C. } 3\pi}$ D. 6π E. NOTA

Solution: It is important to note the property

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}$$

If you combine the desired sum into one fraction you get

$$\frac{\arcsin(x) + \arccos(x)}{\arcsin(x)\arccos(x)} = \frac{\frac{\pi}{2}}{\frac{1}{6}} = 3\pi$$

---

17. Let $V_1, V_2, \ldots, V_{11}$ be the vertices of an 11-gon, and let $\theta_1, \theta_2, \ldots, \theta_{11}$ be the corresponding angles at each vertex. What is the minimum number of positive elements in $\{\cos(\theta_1), \cos(\theta_2), \ldots, \cos(\theta_{11})\}$?

$\boxed{\text{A. } 3}$ B. 4 C. 5 D. 6 E. NOTA

Solution: This problem is really just asking for the maximum number of acute angles possible in an 11-gon. Note that if you can have 3 acute angles of 68 degrees and 8 obtuse angles of 177 degrees (if we are nitpicking about the definition of a set, we can make our acute angles 67; 68; and 69 as well). However, it is not possible to have 4 acute angles because these can sum to at most 360 degrees, and with 7 remaining angles having to sum to more than 1260 degrees. Thus the maximum is 3.

---

18. If $\sec(\theta) + \tan(\theta) = 3$, for some angle $\theta \in \left[0, \frac{\pi}{2}\right)$, compute the value of

$$\sec^3(\theta) + \tan^3(\theta)$$

$\boxed{\text{A. } 7}$ B. 11 C. 15 D. 19 E. NOTA

Solution: We can multiply both sides of the original equation by $\sec(\theta) - \tan(\theta)$ and apply the Pythagorean Identity to get

$$3\sec(\theta) - 3\tan(\theta) = 1$$

Using this equation and the given equation, we get $\sec(\theta) = \frac{5}{3}$ and $\tan(\theta) = \frac{4}{3}$, thus the value of

$$\sec^3(\theta) + \tan^3(\theta) = \frac{189}{27} = 7$$

---

19. The polar curve $r = 3\sin^3(\theta) - 2\sin(\theta)$ is plotted in the Cartesian coordinate system. What is the smallest (least in numerical value) $y-$coordinate of any point on this curve?

A. $-\dfrac{2}{3}$ B. $-\dfrac{1}{2}$ $\boxed{\text{C. } -\dfrac{1}{3}}$ D. $-\dfrac{1}{6}$ E. NOTA

Solution: Note that we are trying to minimize y and since $y = r\sin(\theta)$, we can multiply both sides of the given equation by $\sin(\theta)$ to get

$$y = 3\sin^4(\theta) - 2\sin^2(\theta)$$

which is a quadratic in $\sin^2(\theta)$, minimized when $\sin^2(\theta) = \frac{1}{3}$. Pluggin back in, the lowest y coordinate is $-\frac{1}{3}$.

---

20. When plotted in Cartesian coordinates, how many quadrants does the graph of the function $f(x) = \sin^3(x)\cos^2(x) - \sin^2(x)\cos^3(x)$ pass through?

A. 1 B. 2 C. 3 $\boxed{\text{D. } 4}$ E. NOTA

Solution: Factor the RHS to

$$f(x) = [\sin^2(x)\cos^2(x)](\sin(x) - \cos(x))$$

This is essentially $g(x) = \sin(x) - \cos(x)$ scaled by a nonnegative number $[\sin(x)\cos(x)]$, and $g(x)$ clearly passes through all quadrants.

---

21. If $\ln(\sin(x)) = \frac{3}{5}$ and x is a real number, then what is the value of $\ln(\sin(2x))$?

A. $\dfrac{6}{25}\ln(2)$ B. $\dfrac{18}{25}$ C. $\dfrac{12}{25}\ln(2)$ D. $\dfrac{24}{25}$ $\boxed{\text{E. NOTA}}$

Solution: Note that there is no real value of x that satisfies $\ln(|\sin(x)|) = \frac{3}{5}$ because $e^{\frac{3}{5}} > 1$ is outside of the range of sine, so the desired quantity cannot exist either.

---

22. Let $\sin(x)$, $\cos(x)$, and a be the side lengths of a right triangle for $x, a \in \mathbb{R}$ and $a \in (0, 1)$. Which of the following gives an expression for the value of a in terms of x?

A. $2\sin^2(x) - 1$ B. $2\cos^2(x) - 1$ C. $2\sin^2(x)$ D. $2\cos^2(x)$ $\boxed{\text{E. NOTA}}$

Solution: Because $a \neq 1$ by the given info and $\sin^2(x) + \cos^2(x) = 1$, a cannot be the hypotenuse. Using the Pythagrean Theorem we have that $a^2 + \sin^2(x) = \cos^2(x)$ or $a^2 + \cos^2(x) = \sin^2(x)$. When solving for a, the answer must have a square root, and none of the given answers do.

---

23. In triangle ABC, let BD be the angle bisector of angle ABC, and let R_A and R_C be the lengths of the circumradii of triangles ABD and BDC respectively. If $AB = 6$ and $AC = 12$, compute the value of $\frac{R_A}{R_B}$.

A. $\dfrac{\sqrt{5}}{6}$ $\boxed{\text{B. } \dfrac{1}{2}}$ C. $\dfrac{\sqrt{5}}{3}$ D. $\dfrac{5}{6}$ E. NOTA

Solution: By the Angle Bisector Theorem, we have that

$$\frac{BD}{DC} = \frac{1}{2}$$

Because andles BAD and CAD must be equall in measure, we have by the Extended Law of Sines that the ratio of the circumradii of the two triangles must be $\frac{1}{2}$ also.

---

24. If $\theta \in \left(0, \frac{\pi}{2}\right)$ is a first quadrant angle such that $\sin(\theta) = \frac{2\sqrt{5}}{5}$, then compute the value of

$$\sin\left(2\theta + \arctan\left(\frac{3}{4}\right)\right)$$

$\boxed{\text{A. } 0}$ B. $\dfrac{12}{25}$ C. $\dfrac{24}{25}$ D. 1 E. NOTA

Solution: Using sine angle addition, we have

$$\sin\left(2\theta + \arctan\left(\frac{3}{4}\right)\right) = \frac{4}{5}\sin(2\theta) + \frac{3}{5}\cos(2\theta)$$

Using double angle formulas, we get that $\sin(2\theta) = \frac{4}{5}$ and $\cos(2\theta) = -\frac{3}{5}$. Plugging these back in we get 0

For questions $25 - 27$, for a real-valued input θ, let the counterclockwise rotation matrix in the Cartesian Plane is defined as

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

25. Compute the determinant of $-4R\left(\frac{5\pi}{12}\right)$

A. -4 B. 0 C. 4 $\boxed{\text{D. } 16}$ E. NOTA

Solution: The determinant of any $R(\theta)$ is 1, as you will get the Pythagorean Identity. If you scale a matrix by a constant, you scale its determinant by the constant raised to the dimension of the matrix, 2 in this case. This makes the answer $4^2 = 16$

---

26. Let $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a vector in the Cartesian plane. Compute the sum of the entries of the matrix product

$$\left[R\left(\frac{3\pi}{4}\right)\right]^6 \vec{v}$$

A. $-\sqrt{2}$ B. $\boxed{-1}$ C. 1 D. $\sqrt{2}$ E. NOTA

Solution: Seeing as the rotation matrix rotates a point by a certain angle, applying a rotation six times is simply a rotation by six times the angle! Therefore, we have

$$\left[R\left(\frac{3\pi}{4}\right)\right]^6 = R\left(\frac{9\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then apply this matrix to $\vec{v}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \Rightarrow -3 + 2 = -1$$

---

27. An eigenvector $\vec{x}$ of a matrix A is a special vector that satisfies the matrix equation

$$A\vec{x} = \lambda\vec{x}$$

for some $\lambda \in \mathbb{R}$. A geometric interpretation of this idea is the eigenvectors of a matrix A is the set of all vectors whose image under this multiplication by A is a vector parallel to its pre-image. Nothing these definitions, how many distinct real eigenvectors does the martrix $R\left(\frac{3\pi}{4}\right)$ have?

A. 0 B. $\boxed{1}$
C. 2 D. infinitely many E. NOTA

Solution: When rotating by $\frac{3\pi}{4}$, there is no way to land a nonzero vector back onto the line going through it and the origin. However, the 0 vector turns out to be an eigenvector for all matrices using this definition, so the answer is 1.

---

28. In triangle ABC with right angle at B, $AB = 13$ and $AC = 16$. In which of the following intervals does the $m\angle ACB$ lie?

A. $\left(\frac{\pi}{6}, \frac{\pi}{5}\right)$ B. $\left(\frac{\pi}{5}, \frac{\pi}{4}\right)$ C. $\boxed{\left(\frac{\pi}{4}, \frac{\pi}{3}\right)}$ D. $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ E. NOTA

Solution: To solve this, note we can look to bound the sine of each endpoint angle. We want to find an interval that contains $\arcsin\left(\frac{13}{16}\right)$. We can show that $\frac{13}{16}$ is between $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$ by setting up a compound inequality and squaring all sides and comparing, therefore the interval is

$$\frac{\sqrt{2}}{2} < \frac{13}{16} < \frac{\sqrt{3}}{2} \Rightarrow \arcsin\left(\frac{\sqrt{2}}{2}\right) < \arcsin\left(\frac{13}{16}\right) < \arcsin\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \frac{\pi}{4} < \arcsin\left(\frac{13}{16}\right) < \frac{\pi}{3}$$

---

29. A real number x is generated by selecting a number uniformly and at random from the interval $(0, 2)$. The probability that

$$\frac{1}{3} \leq \tan(x) \leq \frac{1}{2}$$

is p. What is the value of $\tan(p)$?

$\boxed{\text{A. } 5\sqrt{2} - 7}$ B. $\frac{1}{7}$ C. $\frac{1}{6}$ D. $3 - 2\sqrt{2}$ E. NOTA

Solution: We want to look for the probability that the number we generate satisfies

$$\arctan\left(\frac{1}{3}\right) \leq x \leq \arctan\left(\frac{1}{2}\right)$$

The size of this interval is $\arctan(\frac{1}{2}) - \arctan(\frac{1}{3}) = \arctan(\frac{1}{7})$ which you can check by tangent addition. The probability is this number divided by 2 (as the we are uniformly selecting over the interval $(0, 2)$). Using the tangent half angle formula, we have

$$\tan\left(\frac{\arctan(\frac{1}{7})}{2}\right) = \frac{\sin(\arctan(\frac{1}{7}))}{1 + \cos(\arctan(\frac{1}{7}))} = 5\sqrt{2} - 7$$

---

30. You will learn in integral calculus that

$$\int \cot(x)dx = \ln(|\sin(x)|) + C$$

for an arbitrary real constant C. What is the range of the function $f(x) = \ln(|\sin(x)|)$?

A. $(-\infty, \infty)$ $\boxed{\text{B. } (-\infty, 0]}$ C. $[-1, 1]$ D. $[0, \infty)$ E. NOTA

Solution: $|\sin(x)|$ has range $[0, 1]$. The log of this range gives $(-\infty, 0]$