For this test, E. NOTA means "None Of These Answers". All inverse trigonometric functions are restricted to their traditional domains and ranges.

1. Let θ be an angle satisfying
$$\sin^6(\theta) + \cos^6(\theta) = \frac{2021}{2022}$$
What is the value of $\sin^2(\theta) + \cos^2(\theta)$?

A. $\frac{2020}{6063}$ B. $\frac{1}{3}$ C. $\frac{2020}{2021}$ D. 1 E. NOTA

2. The espression $\sin(\theta) + 2\cos(\theta)$ can also be written in the form $-\sqrt{5}\sin(\theta + \phi)$, for some $\phi \in [0, 2\pi)$. What is the value of $\tan(\phi)$?

A. -2 B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. 2 E. NOTA

3. Which of the following quadratic functions have roots $\cos(\theta)$ and $\sin(\theta)$ for some angle $\theta \in [0, 2\pi)$?

A. $f(x) = x^2 - x - 1$ B. $f(x) = x^2 + x + 1$
C. $f(x) = 8x^2 - 4x - 3$ D. $f(x) = 8x^2 - 4x + 3$ E. NOTA

4. The point $(4, 2)$ is rotated $\frac{\pi}{3}$ radians clockwise around the origin, then reflected over the $y-$axis. The result of these transformations is the point (a, b). Compute the value of $|ab|$.

A. $3\sqrt{3} - 4$ B. $3\sqrt{3} + 4$ C. $6\sqrt{3}$ D. $6\sqrt{3} + 3$ E. NOTA

5. The curve
$$y = x + \frac{\sin(2x)}{4}$$
along with the line $x = \pi$ and the $x-$axis bound a region in the $xy-$plane. What is the area of this region?

A. $\frac{\pi^2}{4} + \frac{\pi}{2}$ B. $\frac{\pi^2}{2}$ C. $\frac{\pi^2}{2} + \frac{\pi}{8}$ D. $\frac{\pi^2}{4} + \pi$ E. NOTA

6. For how many angles $\theta \in [0, 2\pi)$ is the following matrix singular?
$$\begin{bmatrix} \sin(\theta) & 3\cos(\theta) \\ \cos(\theta) & 2\sin(\theta) \end{bmatrix}$$

A. 0 B. 2 C. 4 D. 8 E. NOTA

7. What is the area enclosed by the graph of $r = \sin(\theta) + 3\cos(\theta)$ in the polar plane?

A. $\frac{5\pi}{2}$ B. 3π C. $\frac{9\pi}{2}$ D. 5π E. NOTA

8. Quadrilateral $ABCD$ satisfies $AD = 3$, $BC = 4$, $AB = 6$, with $m\angle DAB = m\angle ABC = \frac{\pi}{3}$. If DC has length x, compute x^2.

A. 5 B. 6 C. 7 D. 8 E. NOTA

9. Let $\theta \in [0, 2\pi)$ satisfy

$$\sin^4(\theta) + \cos^4(\theta) = \frac{2}{3}$$

Compute the value of $\sin^8(\theta) + \cos^8(\theta)$.

A. $\frac{1}{3}$ B. $\frac{7}{18}$ C. $\frac{4}{9}$ D. $\frac{1}{2}$ E. NOTA

10. For a real number x, compute the minimum possible value of the expression

$$\sin(x) - 2\cos^2(x)$$

A. $-\frac{17}{8}$ B. -2 C. $-\frac{15}{8}$ D. $-\frac{3}{2}$ E. NOTA

11. For how many real angles $\theta \in [0, 2\pi)$ is the following equation satisfied?

$$\sum_{n=0}^{\infty} \sin(\theta)(-\cos(\theta))^n = \sqrt{2}$$

A. 0 B. 1 C. 2 D. 3 E. NOTA

12. An isosceles triangle ABC has vertex at B with $m\angle B = \phi$ and $\sin(\phi) = \frac{2}{5}$. If $AC = 4$, what is the area of the circumcircle of ABC?

A. 25π B. 32π C. 40π D. 50π E. NOTA

13. Jeffrey's triangle JLU has the interesting property that its area satisfies

$$A(\theta) = \sin(2\theta)$$

where $A(\theta)$ is the area function which depends on $\theta = m\angle JLU$. If $JL = \sqrt{2}$ and $LU = \sqrt{3}$, then what is the length of UJ?

A. 1 B. $\sqrt{2}$ C. $\sqrt{3}$ D. 2 E. NOTA

14. What is the distance between the points $\left(2, \frac{2\pi}{3}\right)$ and $\left(4, \frac{\pi}{6}\right)$, which are both in polar coordinates?

A. $\sqrt{10}$ B. $2\sqrt{3}$ C. $3\sqrt{2}$ D. $2\sqrt{5}$ E. NOTA

15. If $\tan(x) = \frac{1}{2}$, what is the sum of all possible distinct values of $\tan(3x)$?

A. 0 B. $\frac{7}{2}$ C. $\frac{9}{2}$ D. $\frac{11}{2}$ E. NOTA

16. Given that

$$\arcsin(x)\arccos(x) = \frac{1}{6}$$

compute the value of

$$\frac{1}{\arcsin(x)} + \frac{1}{\arccos(x)}$$

A. 3 B. 6 C. 3π D. 6π E. NOTA

17. Let $V_1, V_2, \ldots, V_{11}$ be the vertices of an 11-gon, and let $\theta_1, \theta_2, \ldots, \theta_{11}$ be the corresponding angles at each vertex. What is the minimum number of positive elements in $\{\cos(\theta_1), \cos(\theta_2), \ldots, \cos(\theta_{11})\}$?

A. 3 B. 4 C. 5 D. 6 E. NOTA

18. If $\sec(\theta) + \tan(\theta) = 3$, for some angle $\theta \in \left[0, \frac{\pi}{2}\right)$, compute the value of

$$\sec^3(\theta) + \tan^3(\theta)$$

A. 7 B. 11 C. 15 D. 19 E. NOTA

19. The polar curve $r = 3\sin^3(\theta) - 2\sin(\theta)$ is plotted in the Cartesian coordinate system. What is the smallest (least in numerical value) $y-$coordinate of any point on this curve?

A. $-\frac{2}{3}$ B. $-\frac{1}{2}$ C. $-\frac{1}{3}$ D. $-\frac{1}{6}$ E. NOTA

20. When plotted in Cartesian coordinates, how many quadrants does the graph of the function $f(x) = \sin^3(x)\cos^2(x) - \sin^2(x)\cos^3(x)$ pass through?

A. 1 B. 2 C. 3 D. 4 E. NOTA

21. If $\ln(\sin(x)) = \frac{3}{5}$ and x is a real number, then what is the value of $\ln(\sin(2x))$?

A. $\frac{6}{25}\ln(2)$ B. $\frac{18}{25}$ C. $\frac{12}{25}\ln(2)$ D. $\frac{24}{25}$ E. NOTA

22. Let $\sin(x)$, $\cos(x)$, and a be the side lengths of a right triangle for $x, a \in \mathbb{R}$ and $a \in (0,1)$. Which of the following gives an expression for the value of a in terms of x?

A. $2\sin^2(x) - 1$ B. $2\cos^2(x) - 1$ C. $2\sin^2(x)$ D. $2\cos^2(x)$ E. NOTA

23. In triangle ABC, let BD be the angle bisector of angle ABC, and let R_A and R_C be the lengths of the circumradii of triangles ABD and BDC respectively. If $AB = 6$ and $AC = 12$, compute the value of $\frac{R_A}{R_B}$.

A. $\frac{\sqrt{5}}{6}$ B. $\frac{1}{2}$ C. $\frac{\sqrt{5}}{3}$ D. $\frac{5}{6}$ E. NOTA

24. If $\theta \in \left(0, \frac{\pi}{2}\right)$ is a first quadrant angle such that $\sin(\theta) = \frac{2\sqrt{5}}{5}$, then compute the value of

$$\sin\left(2\theta + \arctan\left(\frac{3}{4}\right)\right)$$

A. 0 B. $\frac{12}{25}$ C. $\frac{24}{25}$ D. 1 E. NOTA

For questions $25 - 27$, for a real-valued input θ, let the counterclockwise rotation matrix in the Cartesian Plane is defined as

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

25. Compute the determinant of $-4R\left(\frac{5\pi}{12}\right)$

A. -4 B. 0 C. 4 D. 16 E. NOTA

26. Let $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a vector in the Cartesian plane. Compute the sum of the entries of the matrix product

$$\left[R\left(\frac{3\pi}{4}\right)\right]^6 \vec{v}$$

A. $-\sqrt{2}$ B. -1 C. 1 D. $\sqrt{2}$ E. NOTA

27. An eigenvector $\vec{x}$ of a matrix A is a special vector that satisfies the matrix equation

$$A\vec{x} = \lambda\vec{x}$$

for some $\lambda \in \mathbb{R}$. A geometric interpretation of this idea is the eigenvectors of a matrix A is the set of all vectors whose image under this multiplication by A is a vector parallel to its pre-image. Nothing these definitions, how many distinct real eigenvectors does the martrix $R\left(\frac{3\pi}{4}\right)$ have?

A. 0 B. 1
C. 2 D. infinitely many E. NOTA

28. In triangle ABC with right angle at B, $AB = 13$ and $AC = 16$. In which of the following intervals does the $m\angle ACB$ lie?

A. $\left(\frac{\pi}{6}, \frac{\pi}{5}\right)$ B. $\left(\frac{\pi}{5}, \frac{\pi}{4}\right)$ C. $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ D. $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ E. NOTA

29. A real number x is generated by selecting a number uniformly and at random from the interval $(0, 2)$. The probability that

$$\frac{1}{3} \leq \tan(x) \leq \frac{1}{2}$$

is p. What is the value of $\tan(p)$?

A. $5\sqrt{2} - 7$ B. $\frac{1}{7}$ C. $\frac{1}{6}$ D. $3 - 2\sqrt{2}$ E. NOTA

30. You will learn in integral calculus that

$$\int \cot(x)dx = \ln(|\sin(x)|) + C$$

for an arbitrary real constant C. What is the range of the function $f(x) = \ln(|\sin(x)|)$?

A. $(-\infty, \infty)$ B. $(-\infty, 0]$ C. $[-1, 1]$ D. $[0, \infty)$ E. NOTA