2022 Nationals Precalculus Hustle Answers and Solutions

1. $\mathbf{\frac{7}{10}}$	**2.** $\frac{7}{3}$	**3.** 18	**4.** -32	**5.** $-\frac{1}{4}$
6. $(-8,8)$	**7.** 672	**8.** $\mathbf{\frac{7}{3}}$	**9.** 5π	**10.** $\frac{3}{4}$
11. $\frac{3}{2}$	**12.** 12	**13.** $\sec(x)$	**14.** -4	**15.** $\frac{4}{7}$
16. 2	**17.** -6	**18.** 230	**19.** -5	**20.** 1
21. 2	**22.** 8	**23.** 10π	**24.** $\frac{7}{25}$	**25.** 5

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1. Find the value of the sum: $\sum_{n=1}^{\infty} \frac{1^n - 2^n + 3^n}{6^n}$.

Ans: $\frac{7}{10}$

Solution: $\sum_{n=1}^{\infty} \frac{1^n - 2^n + 3^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

Each is the sum of a convergent infinite geometric series.

$= \frac{\frac{1}{6}}{1-\frac{1}{6}} - \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{5} - \frac{1}{2} + 1 = \frac{7}{10}$.

2. Suppose $8^{a_1} = 9, 9^{a_2} = 10, \ldots, 127^{a_{120}} = 128$. Find the product $a_1 a_2 a_3 \ldots a_{120}$.

Ans: $\frac{7}{3}$

Solution: $a_1 = \log_8 9, a_2 = \log_9 10, \ldots, a_{120} = \log_{127} 128$.

So $a_1 a_2 a_3 \ldots a_{120} = \log_8 9 \log_9 10 \log_{10} 11 \ldots . \log_{127} 128$.

By Change of Base, $a_1 a_2 a_3 \ldots a_{120} = \frac{\ln(9)}{\ln(8)} \cdot \frac{\ln(10)}{\ln(9)} \cdot \frac{\ln(11)}{\ln(10)} \cdot \ldots \cdot \frac{\ln(128)}{\ln(127)} = \frac{\ln(128)}{\ln(8)} = \log_8 128 = \frac{7}{3}$

3. Find the area interior to the parallelogram with side lengths $2\sqrt{6}$ and $3\sqrt{2}$ and interior angle $60°$.

Ans: 18

Solution: Break the parallelogram into two congruent triangles with side lengths $2\sqrt{6}$ and $3\sqrt{2}$ and interior angle $60°$. The area of one triangle is $\frac{1}{2}(2\sqrt{6})(3\sqrt{2})\sin(60°) = 9$. So the area of the parallelogram is 18.

4. Find the product of all values of *x* that make the matrix singular: $\begin{bmatrix} x+2 & 9 \\ 4 & x+2 \end{bmatrix}$.

Ans: -32.

Solution: The determinant $(x+2)^2 - 4 \cdot 9 = 0$ so $x + 2 = \pm 6$. We see $x = 4, -8$ whose product is -32.

5. Find the product of the slopes of the asymptotes of the graph of $x(t) = 2\tan(t)$, $y(t) = \sec(t)$.

Ans: $-\frac{1}{4}$

Solution: $x^2 = 4\tan^2 t, y^2 = \sec^2 t$. Since $1 + \tan^2 t = \sec^2 t$, $1 + \frac{x^2}{4} = y^2$ so $y^2 - \frac{x^2}{4} = 1$ which is the graph of a vertical hyperbola with asymptotes $y = \pm\frac{1}{2}x$. The product of the slopes of the asymptotes is $\frac{1}{2} \cdot -\frac{1}{2} = -\frac{1}{4}$.

6. Find the domain in interval notation: $f(x) = \log_{2020}\left(4 - |\frac{x}{2}|\right)$.

Ans: $(-8, 8)$

Solution: We require $4 - \left|\frac{x}{2}\right| > 0$ so $\left|\frac{x}{2}\right| < 4$ implying $|x| < 8$. This is the interval $(-8, 8)$.

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7. Find the constant term in the expansion $\left(2x^2 - \frac{1}{x}\right)^9$.
Ans: 672
Solution: $\left(2x^2 - \frac{1}{x}\right)^9 = \binom{9}{3}(2x^2)^3\left(-\frac{1}{x}\right)^6 = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} 2^3 x^6 \cdot \frac{1}{x^6} = 672$

8. The side lengths of a triangle are in arithmetic progression. If its largest angle is $120°$, find the ratio of the largest to smallest sides.
Ans: $\frac{7}{3}$
Solution: Let the side lengths be $x - d, x, x + d$ with $d > 0$.
Then $(x + d)^2 = x^2 + (x - d)^2 - 2x(x - d)\cos(120°)$ by Law of Cosines.
Expanding, $x^2 + 2xd + d^2 = x^2 + x^2 - 2xd + d^2 - 2x(x - d)(-\frac{1}{2})$
so $4xd = x^2 + x(x - d)$ and $5d = 2x$. Letting *x* = 5, *d* = 2, the side lengths are 3, 5, 7. The desired ratio is $\frac{7}{3}$.

9. Find the sum of solutions on $[0, 2\pi)$ to $2\sin(2x) + 1 = 0$.
Ans: 5π
Solution: $\sin(2x) = -\frac{1}{2}$ so $2x = \frac{7\pi}{6} + 2\pi n$ or $2x = \frac{11\pi}{6} + 2\pi n$ for any integer *n*. Dividing by 2, $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$ over the given interval. The sum is $\frac{(7+11+19+23)\pi}{12} = 5\pi$

10. Let $\sin t - \cos t = \frac{1}{2}$. Find $\sin(2t)$.
Ans: $\frac{3}{4}$
Solution: Square both sides: $\sin^2 t - 2\sin t \cos t + \cos^2 t = \frac{1}{4}$. So $1 - \sin(2t) = \frac{1}{4}$ and thus $\sin(2t) = \frac{3}{4}$.

11. Evaluate $\lim_{x\to\infty} \frac{(2x-1)(3x+1)^2}{(2x-1)^2(3x+1)}$.
Ans: $\frac{3}{2}$
Solution: $\lim_{x\to\infty} \frac{3x+1}{2x-1} = \lim_{x\to\infty} \frac{3+1/x}{2-1/x} = \frac{3}{2}$.

12. If $\frac{x^3+5x^2+ax+b}{x+1} = x^2 + cx + 2$ for values *a, b, c,* find the value of $a + b + c$.
Ans: 12
Solution: By synthetic or long division, $\frac{x^3+5x^2+ax+b}{x+1} = x^2 + 4x + (a - 4) + \frac{4+b-a}{x+1}$.
Thus, $4 = c, a - 4 = 2, 4 + b - a = 0$. So *a* = 6, *b* = 2, *c* = 4 and so *a* + *b* + *c* = 12.

13. Where defined, simplify the expression to one single trig function: $\frac{1}{\tan x(\csc x - \sin x)}$
Ans: $\sec x$
Solution: $\frac{1}{\tan x(\csc x - \sin x)} \cdot \frac{\sin x \cos x}{\sin x \cos x} = \frac{\sin x \cos x}{\sin x(1 - \sin^2 x)} = \frac{\cos x}{\cos^2 x} = \sec x$.

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14. $z = \sqrt[6]{2}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$ where $i = \sqrt{-1}$. Let $z^{12} = a + bi$ for real *a, b*. Find *a* + *b*.
Ans: –4
Solution: Using de Moivre's Theorem, $z^{12} = \left(\sqrt[6]{2}\right)^{12}(\cos\frac{12\pi}{8} + i\sin\frac{12\pi}{8}) = 4(0 - i) = 0 - 4i$.
So *a* + *b* = 0 – 4 = –4.

15. Chris owns 4 purple shirts and 4 green shirts. Shirts of the same color are identical. If he selects 2 shirts at random, what is the probability of getting one of each color? Write as a reduced fraction.
Ans: $\frac{4}{7}$
Solution: There are $\binom{8}{2} = 28$ ways to pick 2 shirts. There are $\binom{4}{1}\binom{4}{1} = 4 \cdot 4 = 16$ ways to pick one of each color. The desired probability is $\frac{16}{28} = \frac{4}{7}$.

16. Let $f(x) = \sin(x)$ and $g(x) = 2x$. How many solutions on $[0, 2\pi)$ are there to $(f \circ g)(x) = (g \circ f)(x)$?
Ans: 2
Solution: $\sin(2x) = 2\sin(x)$ so $2\sin(x)\cos(x) = 2\sin(x)$ or $2\sin(x)(1 - \cos(x)) = 0$.
Thus, $x = 0, \pi$, consisting of 2 solutions.

17. Let $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$. Find the dot product $\mathbf{u} \cdot \mathbf{v}$.
Ans: -6
Solution: $\mathbf{u} \cdot \mathbf{v} = 3(2) + (-4)(3) = 6 - 12 = -6$

18. Let *S* be the sum of the first 20 positive odd integers. Let *T* be the sum of the first 20 positive integer multiples of 3. How much more is *T* than *S*?
Ans: 230
Solution: Both are arithmetic series. $S = 1 + 3 + 5 + \cdots + 39 = \frac{1+39}{2}(20) = 400$ while $T = 3 + 6 + 9 + \cdots + 60 = \frac{3+60}{2}(20) = 630$. *T* – *S* = 230.

19. Find the minimum value of $f(x) = 2(3 - 2x)^2 - 5$.
Ans: -5
Solution: Since $2(3 - 2x)^2 \geq 0$, then $2(3 - 2x)^2 - 5 \geq -5$.

20. Let $f(x) = \frac{1+x}{1-x}$. Where defined, if $f\left(f\left(f\left(f(x)\right)\right)\right) = ax + b$ for real numbers *a, b*, find $a + b$.
Ans: 1
Solution: $f\left(f(x)\right) = \frac{1+\frac{1+x}{1-x}}{1-\frac{1+x}{1-x}} = \frac{1-x+1+x}{1-x-(1+x)} = \frac{2}{-2x} = -\frac{1}{x}$. Thus, $f\left(f\left(f\left(f(x)\right)\right)\right) = -\frac{1}{-\frac{1}{x}} = x$.
So *a* = 1 and *b* = 0 with *a* + *b* = 1.

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21. If $4^x = 8 + 2^{x+1}$, find the product of all real solutions.
Ans: 2
Solution: $4^x - 2 \cdot 2^x - 8 = (2^x - 4)(2^x + 2) = 0$ for which $2^x = 4$ or $2^x = -2$. The former is the only one with a real solution and *x* = 2.

22. Emily starts with the number 100 and Barb starts with the number 1. At each turn, Emily adds 10 to her number and Barb doubles her number. What is the least number of turns needed for Barb's number to be greater than or equal to Emily's?
Ans: 8
Solution: At the end of turn *n,* Emily's number is $100 + 10n$ while Barb's number is 2^n . We require $100 + 10n \leq 2^n$. When $n = 7$, Emily has 170 while Barb has 128. However, when $n = 8$, Emily has 180 while Barb has 256. 8 turns are required.

23. Find the exact area enclosed by the polar curve $r = 2\sin\theta - 6\cos\theta$.
Ans: 10π
Solution: Multiply by *r*: $r^2 = 2\,r\sin\theta - 6\,r\cos\theta$ so $x^2 + y^2 = 2y - 6x$. By completing the square, $(x+3)^2 + (y-1)^2 = 10$ so the radius is $\sqrt{10}$. The area is $\pi\sqrt{10}^2 = 10\pi$.

24. Evaluate $\cos(2\,\mathrm{Arctan}(\frac{3}{4}))$.
Ans: $\frac{7}{25}$
Solution: $\cos(2\,\mathrm{Arctan}(\frac{3}{4})) = 2\cos^2(\mathrm{Arctan}(\frac{3}{4})) - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$.

25. Find the maximum value of $f(x) = 3\sin(x) - 4\cos(x)$.
Ans: 5
Solution: $f(x) = 5(\frac{3}{5}\sin(x) - \frac{4}{5}\cos(x)) = 5(\sin(x)\cos(y) - \cos(x)\sin(y))$ where $\cos(y) = \frac{3}{5}$ and $\sin(y) = \frac{4}{5}$. Thus, $f(x) = 5\sin(x-y)$ by an identity, so the maximum of *f* is 5.