For each of the following questions, E) NOTA should be answered when None Of The Answers listed are correct. Throughout, $f^{(n)}(x)$ represents the n^{th} derivative with respect to x, and the 0^{th} derivative is the function itself. Good luck and Have Fun!

(1) If $xy' = y$ and $y(20) = 22$, how many possible values are there for $y(2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

(2) If $xy' = y$ and $y(20) = 22$, how many possible values are there for $y(-2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

(3) If $xy' = y\sin(x)$ and $y(20) = 22$, how many possible values are there for $y(-2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

(4) If $yy' = -x$ and $y(20) = 22$, how many possible values are there for $y(2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

(5) Find $y(2)$ if $y' = 3x^2 + 2$ and $y(1) = 5$.

(a) 8 (b) 10 (c) 12

(d) 14 (e) NOTA

(6) Find $y(2\sqrt{2})$ if $y' = x\sqrt{x^2 + 8}$ and $y(1) = 5$.

(a) $\frac{47}{2}$ (b) $\frac{81}{2}$ (c) $\frac{52}{3}$

(d) $\frac{76}{3}$ (e) NOTA

(7) Find $y(1)$ if $y' = 22xe^{y}$ and $y(0) = \ln\left(\frac{1}{22}\right)$.

(a) $-\ln(44)$ (b) $-\ln(22)$ (c) $-\ln(11)$

(d) $-\ln(5.5)$ (e) NOTA

(8) Find $y(2)$ if $y' = 2xy - 2x - y + 1$ and $y(1) = 5$.

(a) $1 + 4e^{2}$ (b) $1 - 4e^{2}$ (c) $4e^{2} - 1$

(d) $-4e^{2} - 1$ (e) NOTA

(9) A certain population $P(t)$ of animals is governed by the differential equation $\frac{dP}{dt} = kP(2022 - P)$. If a population with an initial number of 20 animals has 22 animals after one year, find which of the following is closest to $\lim_{t\to\infty} P(t)$.

(a) 0 (b) 20 (c) 22

(d) 2022 (e) NOTA

(10) A 4-lb roast, initially at 50° F, is placed in a 400° F oven at 5:00 P.M. After 50 minutes it is found that the temperature $T(t)$ of the roast is 150° F. When will the roast be 200° F? Assume the system obeys Newton's Law of Cooling.

(a) $\frac{50\ln\left(\frac{4}{7}\right)}{\ln\left(\frac{5}{7}\right)}$ (b) $\frac{50\ln\left(\frac{5}{7}\right)}{\ln\left(\frac{4}{7}\right)}$ (c) $\frac{25\ln\left(\frac{4}{7}\right)}{2\ln\left(\frac{5}{7}\right)}$

(d) $\frac{25\ln\left(\frac{5}{7}\right)}{2\ln\left(\frac{4}{7}\right)}$ (e) NOTA

(11) Torricelli's Law states that if a tank with liquid of volume V is draining from a hole in the bottom with area a, then the depth of the water at time t, $y(t)$, is governed by the differential equation $\frac{dV}{dt} = -a\sqrt{2gy}$. Assume $g = 32\ ft/s^{2}$. A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 4 in. is opened in the bottom of the tank. How long will it take (in seconds) for all the water to drain from the tank?

(a) $\frac{168}{5}$ (b) $\frac{224}{5}$ (c) $\frac{336}{5}$

(d) $\frac{672}{5}$ (e) NOTA

(12) Find $y\left(\frac{\pi}{6}\right)$ if $y' + \tan(x)\, y = 2\sec(x)$ and $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$.

(a) $1 + 3\sqrt{2}$ (b) $1 + 2\sqrt{3}$ (c) $1 - 3\sqrt{2}$

(d) $1 - 2\sqrt{3}$ (e) NOTA

(13) A 120 gal tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the perfectly stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt (in lbs) does the tank contain when it is full?

(a) $\frac{6465}{8}$ (b) $\frac{6465}{16}$ (c) $\frac{6465}{32}$

(d) $\frac{6465}{64}$ (e) NOTA

(14) Which of the following is a possible value of y when $x = 2$ if $(3x^2 + 2xy)y' = x^2 + 6xy + 3y^2$ and $(1,2)$ is a point on the graph of the curve defined by this differential equation?

(a) $-3 + \sqrt{93}$ (b) $3 + \sqrt{93}$ (c) $-3 + \sqrt{97}$

(d) $3 + \sqrt{97}$ (e) NOTA

(15) If $xy' - 5x^2y + 3y\ln(y) = 0$, $y > 0$, and $y(1) = e^2$, find $y\left(\frac{1}{2}\right)$.

(a) $e^{27/4}$ (b) $e^{29/4}$ (c) $e^{31/4}$

(d) $e^{33/4}$ (e) NOTA

(16) Which of the following is a solution to the following exact differential equation

$$\left(2xy^3\arctan(xy) + \frac{x^2y^4}{1+x^2y^2}\right) + \left(3x^2y^2\arctan(xy) + \frac{x^3y^3}{1+x^2y^2}\right)\frac{dy}{dx} = 0$$

if $(1,1)$ is a point on the graph of the curve defined by this differential equation?

(a) $x^3y^2\arctan(xy) = \frac{\pi}{4}$ (b) $x^2y^2\arctan(xy) = \frac{\pi}{4}$

(c) $x^2y^3\arctan(xy) = \frac{\pi}{4}$ (d) $x^3y^3\arctan(xy) = \frac{\pi}{4}$ (e) NOTA

(17) Which of the following is a solution to the following differential equation

$$(2x+2y+1)+(3x+3y+1)\frac{dy}{dx}=0$$

if $y(0)=1$?

(a) $\ln(x+y)+2x+3y=3$ (b) $\ln(x+y)+3x+3y=3$
(c) $\ln(x+y)+3x+2y=2$ (d) $\ln(x+y)+2x+2y=2$ (e) NOTA

(18) What is the general form of the solution to $y''-y'-12y=0$? Assume y is a function of x.

(a) $y=C_1e^{2x}+C_2e^{-6x}$ (b) $y=C_1e^{-4x}+C_2e^{3x}$
(c) $y=C_1e^{4x}+C_2e^{-3x}$ (d) $y=C_1e^{-2x}+C_2e^{6x}$ (e) NOTA

(19) What is the general form of the solution to $y''+4y'+13y=0$? Assume y is a function of x.

(a) $y=C_1e^{-2x}\sin(6x)+C_2e^{-2x}\cos(6x)$
(b) $y=C_1e^{2x}\sin(6x)+C_2e^{2x}\cos(6x)$
(c) $y=C_1e^{-5x}+C_2e^{x}$
(d) $y=C_1e^{5x}+C_2e^{-x}$
(e) NOTA

(20) What is the general form of the solution to $y'''+3y''+3y'+y=0$? Assume y is a function of x.

(a) $y=C_1e^{-x}+C_2xe^{-x}+C_3x^2e^{-x}$
(b) $y=C_1e^{x}+C_2xe^{x}+C_3x^2e^{x}$
(c) $y=C_1xe^{-x}+C_2x^2e^{-x}+C_3x^3e^{-x}$
(d) $y=C_1xe^{x}+C_2x^2e^{x}+C_3x^3e^{x}$
(e) NOTA

(21) What is the general form of the solution to $4x^2y''-4xy'+3y=0$? Assume y is a function of x.

(a) $y=C_1e^{\frac{x}{2}}+C_2e^{\frac{3x}{2}}$ (b) $y=C_1e^{-\frac{x}{2}}+C_2e^{-\frac{3x}{2}}$
(c) $y=C_1\sqrt{x}+C_2\sqrt{x^3}$ (d) $y=\frac{C_1}{\sqrt{x}}+\frac{C_2}{\sqrt{x^3}}$ (e) NOTA

(22) What is the general form of the solution to $x^2y'' + 5xy' + 4y = 0$? Assume y is a function of x.

(a) $y = C_1x^2 + C_2x^2\ln(x)$ (b) $y = \frac{C_1}{x^2} + \frac{C_2\ln(x)}{x^2}$
(c) $y = C_1e^{2x} + C_2xe^{2x}$ (d) $y = C_1e^{-2x} + C_2xe^{-2x}$ (e) NOTA

(23) Which of the following is not a possible value of λ which satisfies the boundary value problem below for non-trivial y?

$$x^2y'' + xy' + \lambda y = 0; y(1) = y(e) = 0$$

(a) π^2 (b) $4\pi^2$ (c) $8\pi^2$

(d) $16\pi^2$ (e) NOTA

(24) Given that $y = x$ is a solution to $(x^2 - 1)y'' + xy' - y = 0$, find the solution to this differential equation if $y(\sqrt{2}) = 0$ and $y'(\sqrt{2}) = 1$.

(a) $y = \sqrt{x^2 - 1} - \frac{x}{\sqrt{2}}$ (b) $y = \frac{\sqrt{x^2-1}}{x} - \frac{x}{2}$
(c) $y = \frac{\sqrt{2x^2-2}}{x} - \frac{x}{\sqrt{2}}$ (d) $y = \sqrt{2x^2 - 2} - x$ (e) NOTA

(25) Let $f(x)$ be a continuous, differentiable function satisfying

$$\left(f(x)\right)^2 = \int_0^x \left(f(t) + f'(t)\right)^2 dt + 2022(1 - x)$$

Which of the following is a possible value of $f\left(\frac{\pi}{6}\right)$?

(a) $\frac{\sqrt{674}}{2}$ (b) $-\frac{3\sqrt{2022}}{2}$ (c) $\frac{3\sqrt{674}}{2}$

(d) $-\frac{\sqrt{2022}}{2}$ (e) NOTA

(26) Let y_p be the particular solution to the non-homogenous differential equation

$$y''' + 4y' = 8\sec(2x)$$

Find $y_p(\pi)$.

(a) -2π (b) 2π (c) $-\pi$

(d) π (e) NOTA

(27) Which of the following is equivalent to $4e^{\begin{bmatrix} 0 & \ln(2) \\ \ln(2) & 0 \end{bmatrix}}$?

(a) $\begin{bmatrix} -5 & -3 \\ -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}$ (e) NOTA

(28) What is the radius of convergence of the Maclaurin series of the general solution to the differential equation $(x^2 - 6x + 25)y'' + \tan\left(\frac{x}{2022}\right)y' - y = 0$?

(a) 0 (b) 1011π (c) 2022π

(d) 5 (e) NOTA

Remaining questions are on the next page.

For the last two questions on this test, you are expected to use the important Differential Equations concept of the Laplace Transform: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$. The below table of Laplace Transforms may be helpful:

$f(t)$	$\mathcal{L}\{f(t)\}\,; s > 0$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$
$\int_0^t f(u)du$	$\frac{1}{s}F(s)$
$e^{at}f(t)\,; a \in \mathbb{R}$	$F(s-a)\,;\ s > a$
$u(t-a)f(t-a) = \begin{cases} 0 & ,t < a \\ f(t-a), & t \geq a \end{cases}; a \in \mathbb{R}^+$	$e^{-as}F(s)$
$f(t) * g(t) = \int_0^t f(u)g(t-u)du$	$F(s)G(s)$
$t^n f(t)\,; n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(v)dv$
$t^n\,; n \in \mathbb{Z}$	$\frac{n!}{s^{n+1}}$

(29) Find $f(6)$ if $f'(t) - u(t-1)f'(t-1) = 1$ and $f(0) = 0$.
Hint: Consider $\mathcal{L}\{1 + [\![t]\!]\}$, where $[\![t]\!]$ is the greatest integer less than or equal to t.

(a) 6 (b) 7 (c) 15

(d) 21 (e) NOTA

(30) Find $f(3)$ if

$$4\int_0^t u \cdot f'(u) \cdot f'(t-u)du = 3\int_0^t u \cdot f(u) \cdot f''(t-u)du$$

and $f(0) = f'(0) = 0$ and $f(1) = 1$.

(a) 3 (b) 9 (c) 27

(d) 81 (e) NOTA