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|---|---|---|---|---|
| 1. A | 7. C | 13. D | 19. C | 25. C |
| 2. B | 8. A | 14. D | 20. D | 26. B |
| 3. B | 9. C | 15. C | 21. A | 27. C |
| 4. D | 10. A | 16. A | 22. B | 28. B |
| 5. A | 11. E | 17. C | 23. C | 29. C |
| 6. B | 12. B | 18. A | 24. B | 30. D |

1. Taking the derivative of our function, we get that $y'(x) = 2x$. Thus $y'(1) = 2$. Using point-slope form, $y - 5 = 2(x - 1) \to y = 2x + 3$. Thus the $y$-intercept is at $\boxed{3}$.

2. The slope of the secant line is $\dfrac{f(2) - f(1)}{2 - 1} = 0$. We now calculate $f'(x) = 3x^2 - 10x + 8$. If we set these two quantities equal to each other, we get that the equation $3x^2 - 10x + 8 = 0$ is satisfied when $x = \frac{4}{3}, 2$. However, the Mean Value Theorem states that the $c$ values cannot be the endpoints of the interval. Thus the Mean Value Theorem only guarantees $\boxed{c = \dfrac{4}{3}}$.

3. This looks very similar to a Riemann sum interpretation of an integral, except now the sum goes to $2n$. To address this, we can say that $m = 2n$ which will match the form we are looking for. Thus, our limit is equivalent to $\displaystyle\lim_{m\to\infty} \sum_{i=1}^{m} \frac{m/2}{i^2 + m^2/4} = \lim_{m\to\infty} \frac{1}{2m} \sum_{i=1}^{m} \frac{1}{\frac{i^2}{m^2} + 1/4}$. Now, we can convert this to an integral: $\displaystyle\frac{1}{2}\int_0^1 \frac{dx}{x^2 + \frac{1}{4}} = \frac{1}{2} \cdot 2\arctan 2x]_0^1 = \boxed{\arctan 2}$.

4. We will go through each answer choice. Choice $A$ is wrong since in the case $a = -1$, $b = 1$ and $f(x) = x^2$ the two tangent lines have the same $y$-intercept. Choice $B$ is always wrong since the $x^2$ coefficient is nonzero since $f(x)$ is degree 2. When using the Mean Value Theorem of Integrals on a quadratic, we end up having to solve a cubic equation which could naturally have 3 potential solutions. For choice $D$, say that $f(x) = c_2x^2 + c_1x + c_0$ where $c_2 \neq 0$ and the coefficients are real. Using the Mean Value Theorem for Derivatives, we want to solve $2c_2x + c_1 = \dfrac{c_2(b^2 - a^2) + c_1(b - a)}{b - a}$. The right-hand side of the equation can be simplified to $c_2(b + a) + c_1$ since $b - a \neq 0$. This means our $c$ value solves the equation $2c_2x + c_1 = c_2(b + a) + c_1$. We can now clearly see that this occurs when $x = \dfrac{a + b}{2}$.

5. We want to evaluate $\displaystyle\sum_{k=2}^{20} k(k-1)$ since $1^k = 1$ for all positive integers $k$. This sum can be solved using the Hockey Stick Identity or be realizing that $\displaystyle\sum_{k=2}^{20} k(k-1) = \sum_{k=2}^{20} k^2 - k =$ $\displaystyle\left(\frac{20 \cdot 21 \cdot 41}{6} - 1\right) - \left(\frac{20 \cdot 21}{2} - 1\right) = 2869 - 209 = \boxed{2660}$.

6. This is very similar to the previous question except now $(-1)^k = -1$ if $k$ is an odd integer. This means we want to evaluate $\sum_{k=2}^{20} k(k-1)(-1)^{k-1} = \sum_{i=1}^{10} -(2i)(2i-1) + \sum_{j=1}^{9}(2j+1)(2j)$ where the first sum handles the even integers and the second sum handles the odd integers. Evaluating these like in question 5 gives us the answer is $-4 \cdot \frac{10 \cdot 11 \cdot 21}{6} + 2 \cdot \frac{10 \cdot 11}{2} + 4 \cdot \frac{9 \cdot 10 \cdot 19}{6} + 2 \cdot \frac{9 \cdot 10}{2} = \boxed{-200}$.

7. Write the integral as $\int_0^4 \frac{2\sqrt{4-x}}{2\sqrt{x}} dx$.We do this because now the $u$-substitution $u = \sqrt{x}$ transforms our integral to $\int_0^2 2\sqrt{4-u^2}du = 2 \cdot \frac{1}{4} \cdot 4\pi = \boxed{2\pi}$ since we are finding the area of a quarter circle.

8. To approximate the area bounded by $f(x)$ and the $x$-axis with rectangles of length $\frac{5-1}{4} = 1$, we just need to find $|f(1)| + |f(2)| + |f(3)| + |f(4)| = \ln(2) + \ln\left(\frac{7}{2}\right) + \ln\left(\frac{17}{2}\right) + \ln\left(\frac{31}{2}\right) = \boxed{\ln\left(\frac{3689}{4}\right)}$.

9. Let us find $f'(x)$ first. We see that $f'(x) = e^{2e^{3e^x}} \cdot \frac{d}{dx}2e^{3e^x} = e^{2e^{3e^x}} \cdot 2e^{3e^x} \cdot \frac{d}{dx}3e^x = e^{2e^{3e^x}} \cdot 2e^{3e^x} \cdot 3e^x$. Plugging in $x = \ln(\ln(2))$ gives us $e^{2e^{3\ln 2}} \cdot 2e^{3\ln 2} \cdot 3\ln 2 = e^{2\cdot 8} \cdot 2 \cdot 8 \cdot 3 \ln 2 = \boxed{48e^{16}\ln 2}$.

10. We first need to find $\frac{dy}{dx}$. Taking the derivative of our equation gives us that $y^2 + 2xy\frac{dy}{dx} + 2x + 3y^2\frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{1-2x-y^2}{2xy+3y^2}$. In addition, $\frac{dy}{dx}|_{(1,-1)} = -2$. Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{(2xy+3y^2)(-2-2y\frac{dy}{dx}) - (2y+2x\frac{dy}{dx}+6y\frac{dy}{dx})(1-2x-y^2)}{(2xy+3y^2)^2}$ by the quotient rule. Since we know the values of everything in this equation at the point $(1,-1)$, we get the answer as $\frac{(1)(-6)-(6)(-2)}{1^2} = -6 + 12 = \boxed{6}$.

11. We want to find all $f(x)$ that satisfy $\int_0^1 f(x)dx = \int_0^1 \sqrt{1+(f'(x))^2}dx$. As a start, let's try to find all functions that satisfy $f(x) = \sqrt{1+(f'(x))^2}$ since that would imply both integrals are equal. Squaring both sides and differentiating with respect to $x$ gives us that $2f(x)f'(x) = 2f'(x)f''(x)$. Thus $f'(x) = 0$ or $f(x) = f''(x)$. In the first case, $f'(x) = 0 \rightarrow f(x) = c > 0$. Plugging this into our original equation gives us that $c = \sqrt{1+0^2} = 1$. Thus $f(x) = 1$ is our first function that works. Now onto the second case. If we say that $f(x) = e^{rx}$, then $f''(x) - f(x) = 0$ is equivalent to finding the $r$ values that solve $r^2 - 1 = 0$. These $r$ values are clearly $\{-1, 1\}$. Thus we get that $f(x) = c_1e^x + c_2e^{-x}$ satisfies our differential equation. We now must plug it back in,

$(c_1e^x + c_2e^{-x})^2 = 1 + (c_1e^x - c_2e^{-x})^2 \to c_1c_2 = \dfrac{1}{4}$. Thus we have infinitely many functions that work of the form $f(x) = ae^x + \dfrac{1}{4a}e^{-x}$ where $a \in \mathbb{R}^+$ (to ensure that $f(x)$ is never negative). Finally, the answer is None Of The Above since there are infinitely many functions.

12. The formula for arc length over the given interval is $\displaystyle\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}$. However, using the half-angle formula, we see that our integral is equivalent to $\displaystyle 2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}dt = 2\int_0^{2\pi} \sin\frac{t}{2}dt = -4\cos\frac{t}{2}]_0^{2\pi} = 4 + 4 = \boxed{8}$.

13. We want to find the $r$ that satisfies $\frac{d}{dt}(\frac{4}{3}\pi r^3) = \frac{d}{dt}(4\pi r^2)$. Computing this gives us that $4\pi r^2\frac{dr}{dt} = 8\pi r\frac{dr}{dt}$. Since $\frac{dr}{dt}$ is a positive constant, we can divide it from both sides and simplify to get that $\boxed{r = 2}$.

14. Given the information in the problem statement, we know that $\det(M(x, x^2, x^3)) = (x^3 - x)(x^3 - x^2)(x^2 - x)$. Taking the derivative with the product rule gives us

$$\frac{d}{dx}[\det(M)] = (x^3-x)(x^3-x^2)(2x-1)+(x^3-x)(3x^2-2x)(x^2-x)+(3x^2-1)(x^3-x^2)(x^2-x).$$

Plugging in $x = 2$ gives us $(6)(4)(3) + (6)(8)(2) + (11)(4)(2) = 72 + 96 + 88 = \boxed{256}$.

15. Using the disk method, our volume is simply $\displaystyle\pi\int_0^1 \ln^4(x)dx$. If we use integration by parts with $u = \ln^4(x)$ and $dv = 1$, then our integral equals $\displaystyle\pi(x\ln^4(x)]_0^1 - \int_0^1 4\ln^3(x)dx) = -4\pi\int_0^1 \ln^3(x)dx$. If we do a very similar process a couple more times, we get that $\displaystyle -4\pi\int_0^1 \ln^3(x)dx = 12\pi\int_0^1 \ln^2(x)dx = -24\pi\int_0^1 \ln(x)dx = -24\pi[x\ln(x) - x]_0^1 = -24\pi(-1) = \boxed{24\pi}$.

16. It is known that $\dfrac{dy}{dx}$ is simply $\dfrac{dy/dt}{dx/dt}$. Thus, the answer is simply $\boxed{\dfrac{\sin t}{1-\cos t}}$.

17. We will use the closed interval method. First, we find that $f'(x) = \dfrac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2} = \dfrac{-x^2+2x+1}{(x^2+1)^2}$. The derivative is equal to 0 when $x \in \{1+\sqrt{2}, 1-\sqrt{2}\}$, however, the larger element is outside the interval $[-1, 1]$ so we can ignore it. Thus we now only need to compare the values of $f(-1), f(1), f(1-\sqrt{2})$. Doing so yields a maximum of 0 at $\boxed{x = 1}$.

18. Let us assume there is a tangent point at $x = x_0$. This would mean that $e^{2x_0} = 2\sqrt{ex_0}$. In addition, the slope of the tangent line at $x = x_0$ must be equal. This would mean that $2e^{2x_0} = \sqrt{\frac{e}{x_0}}$. We can now solve for $x$ since $2e^{2x_0} = 2(e^{2x_0}) = 2(2\sqrt{ex_0}) = 4\sqrt{ex_0} = \sqrt{\frac{e}{x_0}}$. Cross-multiplying and simplifying gives us that $x_0 = \frac{1}{4}$. In addition, the $y$ value associated with this $x$ is $\sqrt{e}$. Finally, this means the answer is $\frac{1}{4} \cdot \sqrt{e} = \boxed{\frac{\sqrt{e}}{4}}$.

19. Let us find a closed form for $f(x)$ first. $f(x) = \sqrt{x + f(x)} \to (f(x))^2 - f(x) = x$. Completing the square gives us that $(f(x) - \frac{1}{2})^2 = x + \frac{1}{4} \to f(x) = \sqrt{x + \frac{1}{4}} + \frac{1}{2}$. This means that the average value of $f(x)$ over the interval $[2, 6]$ is

$$\frac{1}{6-2}\int_2^6 \sqrt{x + \frac{1}{4}} + \frac{1}{2}dx = \frac{1}{4}(\frac{2}{3}(x + \frac{1}{4})^{3/2} + \frac{x}{2}]_2^6).$$

This can be evaluated to $\frac{1}{4}(\frac{125}{12} + 3 - \frac{9}{4} - 1) = \frac{1}{4} \cdot \frac{61}{6} = \boxed{\frac{61}{24}}$.

20. This is a linear differential equation. If we multiply our equation by $e^{\int -1 dx} = e^{-x}$, we get that $e^{-x}(y' - y) = \frac{1}{1+x^2}$. However, notice that the left-hand side of the equation equals $(ye^{-x})'$. Thus if we integrate both sides we get that $ye^{-x} = \arctan x + C$ which means that $y = e^x \arctan(x) + Ce^x$. Since $y(0) = 0$, we find that $C = 0$. Thus, $y = e^x \arctan(x)$ and $y(1) = e^1 \arctan(1) = \boxed{\frac{\pi e}{4}}$.

21. This is a separable equation. We can say that $\frac{dy}{y} = e^x dx$. Integrating both sides means that $\ln|y| = e^x + C$. Plugging in the point $(0, 1)$ means that $C = -1$. Thus $\ln(y) = e^x - 1$. Plugging in $x = 1$ gives us that $\ln(y) = e - 1$ or that $\boxed{y = e^{e-1}}$.

22. We could do calculus to find this volume. However, it is much simpler to notice that the shape we create is a hemisphere of radius 2. The volume of a hemisphere is $\frac{2}{3}\pi r^3$. Thus, our answer is $\frac{2}{3}\pi \cdot 8 = \boxed{\frac{16\pi}{3}}$.

23. We will go through each choice. Choice I diverges because $\sin^2 x > k > 0$ enough of the time to make the integrand look like $\frac{k}{x}$ as $x$ grows large which would diverge. Choice II diverges because at 0, the quantity is undefined and the power of the 0 in the denominator is at least 1. Choice III converges and with a $u$-substitution of $u = \frac{1}{\sqrt{-\ln(x)}}$, the integral turns into a multiple of $\int_0^\infty e^{-x^2} dx$. Choice IV diverges because as $x$ grows large, the integrand looks like $\frac{1}{\sqrt{x}}$ which would diverge. Thus $\boxed{\text{III}}$ is the only convergent integral.

24. While one could use integration by parts and bash this out, the trick here is to realize what is given to us. The integral present is the remainder term of a Maclaurin series for a function that has been approximated to the $x^3$ power evaluated at $x = 1$. Subtracting this from our function means we want to find the cubic approximation for $f(x) = \sin(x)$ at $x = 1$. Since $\sin x \sim x - \frac{x^3}{6}$, this is simply $1 - \frac{1^3}{6} = \boxed{\frac{5}{6}}$.

25. It is pretty well-known that the Maclaurin series of $f(x) = -\ln(1-x)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ when $x \in [-1, 1)$. All we have to do is plug in $x = \frac{1}{2}$ to get that the answer is $-\ln\left(1 - \frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \boxed{\ln(2)}$.

26. First we will start with the sine triple angle formula. Using complex numbers, $\sin(3x) = \text{Im}[(\cos x + i \sin x)^3] = 3\cos^2 x \sin x - \sin^3 x = 3\sin x - 4\sin^3 x$. Thus we can isolate $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$. Now onto the integral.

$$\int_0^\infty \frac{\sin^3 x}{x} dx = \int_0^\infty \frac{3\sin x - \sin 3x}{4x} dx = \frac{3}{4} \cdot \frac{\pi}{2} - \frac{1}{4} \int_0^\infty \frac{\sin 3x}{x} dx$$

Set $u = 3x$. This means that our integral equals $\frac{3\pi}{8} - \frac{1}{4} \int_0^\infty \frac{\sin u}{u} du = \frac{3\pi}{8} - \frac{\pi}{8} = \boxed{\frac{\pi}{4}}$.

27. The $x$-coordinate of the centroid is given by $\frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}$. Doing this gives us the answer is

$$\frac{\int_0^a a^2 x - x^3 dx}{\int_0^a a^2 - x^2 dx} = \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - \frac{a^3}{3}} = \frac{\frac{a^4}{4}}{\frac{2a^3}{3}} = \boxed{\frac{3a}{8}}.$$

28. The error occurs in $\boxed{\text{Line 2}}$. The first step is similar to the steps taken to solve question 26. However, this question differs from question 26 because at $x = 0$, our integral doesn't exist. Thus, while in question 26 we could split the integral into two separate integrals and solve, here, since neither of the two integrals in Line 3 independently exist, we cannot split the integral in the previous line.

29. Let us try to represent the region $y \le x \le 1$ and $0 \le y \le 1$ in a different way to make this integral solvable. Drawing the region quickly yields that this region can also be represented by $0 \le x \le 1$ and $0 \le y \le x$. Thus, by Fubini's Theorem, our integral equals $\int_0^1 \int_0^x \frac{\sin x}{x} dy dx = \int_0^1 \sin x dx = -\cos x]_0^1 = \boxed{1 - \cos 1}$.

30. We have that $\frac{d}{dx} \int_0^1 \frac{e^x}{x} dy = \frac{d}{dx} \frac{e^x}{x} y]_0^1 = \frac{d}{dx} \frac{e^x}{x} = \boxed{\frac{e^x(x-1)}{x^2}}$. This is the answer.