Assume the domain and range of all functions are limited to the real numbers. Don't spend too much time on any one problem. NOTA means "None of the Above."

Good luck, and have fun! ☺

1. What is the y-intercept of the tangent line to the graph $y = x^2 + 4$ at the point $(1, 5)$?

(A) 3 (B) 2 (C) 1 (D) 4 (E) NOTA

2. What is the c value(s) guaranteed by the Mean Value Theorem for Derivatives for the function $f(x) = x^3 - 5x^2 + 8x$ on the interval $[1, 2]$?

(A) $\frac{4}{3}, 2$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5-\sqrt{7}}{2}$ (E) NOTA

3. Compute the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{2n} \frac{n}{i^2 + n^2}$$

(A) $\frac{\pi}{4}$ (B) $\arctan(2)$ (C) $\frac{\pi}{2}$ (D) $\frac{1}{2}\arctan(2)$ (E) NOTA

4. You are given a random polynomial $f(x)$ of degree 2 with real coefficients. Also, let a, b be real numbers such that $a < b$. Which of the following must be true?

(A) The y-intercept of the tangent line to $f(x)$ at $x = a$ can never be equal to the y-intercept of the tangent line to $f(x)$ at $x = b$.

(B) There exists an $x_0 \in (a, b)$ such that $f''(x_0) = 0$.

(C) There is only a single c value that satisfies the Mean Value Theorem for Integrals for $f(x)$ on the interval $[a, b]$.

(D) The c value guaranteed by the Mean Value Theorem for Derivatives for $f(x)$ on the interval $[a, b]$ is $\frac{a+b}{2}$.

(E) NOTA

5. Evaluate $\dfrac{d}{dx}\sum_{k=2}^{20}(k-1)x^k$ at $x = 1$.

(A) 2660 (B) 2056 (C) 1028 (D) 2280 (E) NOTA

6. Now evaluate $\dfrac{d}{dx}\sum_{k=2}^{20}(k-1)x^k$ at $x = -1$.

(A) 200 (B) -200 (C) -380 (D) -360 (E) NOTA

7. Evaluate $\displaystyle\int_0^4 \sqrt{\frac{4-x}{x}}dx$.

(A) $\dfrac{\pi}{2}$ (B) π (C) 2π (D) 4π (E) NOTA

8. Use a Left-Hand Riemann sum approximation to find the area bound by $y = \ln(x^2 - \frac{1}{2})$ and the x-axis from $x = 1$ to $x = 5$ using 4 subdivisions of equal length.

(A) $\ln\left(\dfrac{3689}{4}\right)$ (B) $\ln\left(\dfrac{3689}{8}\right)$ (C) $\ln\left(\dfrac{3689}{16}\right)$ (D) $4\ln\left(\dfrac{3689}{16}\right)$ (E) NOTA

9. Evaluate $f'(\ln(\ln(2)))$ given that $f(x) = e^{2e^{3e^x}}$.

(A) $36e^{12}\ln(2)$ (B) $72\ln^2(2)$ (C) $48e^{16}\ln(2)$ (D) $96e^{16}\ln(2)$ (E) NOTA

10. Find $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$ for $xy^2 + x^2 + y^3 = x$.

(A) 6 (B) -6 (C) -2 (D) -18 (E) NOTA

11. How many differentiable functions $f : (0, 1) \to \mathbb{R}^+$ exist such that the area under f equals the the arc length of f?

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

12. Find the arc length of the curve parameterized by $x = t - \sin(t)$ and $y = 1 - \cos(t)$ for $0 \leq t \leq 2\pi$.

(A) 4 (B) 8 (C) π (D) 2π (E) NOTA

13. A sphere is growing in size such that $\frac{dr}{dt}$ is a positive constant and r is the radius. At what radius is the volume of the sphere growing at the same rate as the surface area of the sphere?

(A) $\sqrt{3}$ (B) 4 (C) 3 (D) 2 (E) NOTA

14. Define the Vandermonde matrix as shown below.$M(x_1, x_2, \ldots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$.

It is known that $\det(M(x_1, x_2, \ldots, x_n)) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$. Calculate $\frac{d}{dx}[\det(M(x, x^2, x^3))]$ evaluated at $x = 2$.

(A) 28 (B) 48 (C) 192 (D) 256 (E) NOTA

15. What is the volume of the solid formed by revolving the region bounded by the graph of $y = \ln^2(x), y = 0, x = 0,$ and $x = 1$ about the x-axis?

(A) 4π (B) πe (C) 24π (D) $12\pi e$ (E) NOTA

16. Find $\frac{dy}{dx}$ of the set of parametric equestions: $x = t - \sin t$ and $y = 1 - \cos t$.

(A) $\frac{\sin t}{1 - \cos t}$

(B) $\frac{1 - \cos t}{\sin t}$

(C) $\frac{\cos t - 1}{\sin t}$

(D) $\frac{t \sin t + 2\cos t - 2}{(t - \sin t)^2}$

(E) NOTA

17. On the interval $[-1, 1]$, where does $f(x) = \frac{x-1}{x^2+1}$ achieve an absolute maximum?

(A) $1-\sqrt{2}$ (B) 0 (C) 1 (D) $1+\sqrt{2}$ (E) NOTA

18. Find the product of the abscissa and ordinate of the point where $y = e^{2x}$ and $y = 2\sqrt{ex}$ are tangent to each other. If no such point exists, the answer is 1337.

(A) $\frac{\sqrt{e}}{4}$ (B) $\frac{\sqrt{e}}{2}$ (C) $\sqrt{e}$ (D) 1337 (E) NOTA

19. Find the average value of $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$ over the interval $[2, 6]$.

(A) $\frac{61}{6}$ (B) $\frac{37}{24}$ (C) $\frac{61}{24}$ (D) $\frac{61}{96}$ (E) NOTA

20. Find $y(1)$ given that $y' - y = \frac{e^x}{1+x^2}$ and $y(0) = 0$.

(A) $\frac{e}{2}$ (B) 1 (C) $\frac{e}{\pi}$ (D) $\frac{\pi e}{4}$ (E) NOTA

21. Now find $y(1)$ given that $y' = ye^x$ and $y(0) = 1$.

(A) e^{e-1} (B) e^e (C) e^{e+1} (D) 0 (E) NOTA

22. Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and has cross sections perpendicular to the x-axis in the shape of semicircles.

(A) $\frac{8\pi}{3}$ (B) $\frac{16\pi}{3}$ (C) $\frac{32\pi}{3}$ (D) 16 (E) NOTA

23. Which of the following integrals converge?

$$\text{I} : \int_0^{\infty} \frac{\sin^2 x}{x} dx$$
$$\text{II} : \int_0^{\infty} x^{-1}e^{-x} dx$$
$$\text{III} : \int_0^{1} \frac{dx}{\sqrt{-\ln(x)}}$$
$$\text{IV} : \int_0^{\infty} \frac{dx}{\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}}$$

(A) I, II (B) II, III, IV (C) III (D) III, IV (E) NOTA

24. Let $f^{(n)}(a)$ represent the nth derivative of f evaluated at $x = a$. Evaluate:

$$f(1) - \frac{1}{6}\int_0^1 f^{(4)}(t)(1-t)^3 dt$$

where $f(x) = \sin x$.

(A) 1 (B) $\frac{5}{6}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$ (E) NOTA

25. Max is writing a Mu Alpha Theta test. The probability that he writes a good test is given by the expression $\sum_{n=1}^{\infty} \frac{1}{n2^n}$. What is this probability?

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\ln 2$ (D) 1 (E) NOTA

26. Evaluate $\int_0^{\infty} \frac{\sin^3 x}{x}$. You may want to use the fact that $\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$.

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{8}$ (D) $\frac{3\pi}{8}$ (E) NOTA

27. Find the x-coordinate of the centroid of the region bound by $y = a^2 - x^2$, $x = 0$, and $y = 0$ in terms of a where a is a positive constant.

(A) $\frac{a}{4}$ (B) $\frac{a}{3}$ (C) $\frac{3a}{8}$ (D) $\frac{3a}{16}$ (E) NOTA

28. In which line does the first error occur when trying to evaluate $I = \int_0^\infty \frac{\sin^3 x}{x^2} dx$?

$$\text{Line 1} : I = \int_0^\infty \frac{3\sin x - \sin 3x}{4x^2} dx.$$

$$\text{Line 2} : I = \frac{3}{4}\int_0^\infty \frac{\sin x}{x^2} dx - \frac{1}{4}\int_0^\infty \frac{\sin 3x}{x^2} dx$$

Set $u = 3x$

$$\text{Line 3} : I = \frac{3}{4}\int_0^\infty \frac{\sin x}{x^2} dx - \frac{3}{4}\int_0^\infty \frac{\sin u}{u^2} du$$

$$\text{Line 4} : I = 0$$

(A) Line 1 (B) Line 2 (C) Line 3 (D) Line 4 (E) NOTA

29. Evaluate the double integral: $\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$

(A) $\sin 1$ (B) $1 - \sin 1$ (C) $1 - \cos 1$ (D) $\cos 1$ (E) NOTA

30. You're at the end of the test! As a reward, evaluate $\frac{d}{dx}\int_0^1 \frac{e^x}{x} dy$.

(A) 0 (B) 1 (C) $\frac{e^x}{x}$ (D) $\frac{e^x(x-1)}{x^2}$ (E) NOTA