All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

∼∼∼∼∼∼∼∼ *Good luck, and have fun!* ∼∼∼∼∼∼∼∼

1) Evaluate using Boolean algebra: $1+1$.

A) 0 B) 1 C) 2 D) 10 E) NOTA

2) Which of the following statements is logically equivalent to "If Daniel reads a book, then Chloe learns calculus"?

A) If Chloe does not learn calculus, then Daniel does not read a book.
B) If Chloe learns calculus, then Daniel reads a book.
C) If Chloe reads a book, then Daniel learns calculus.
D) If Daniel does not read a book, then Chloe does not learn calculus.
E) NOTA

3) A, B, and C are random Boolean values that can be true or false with equal probability. Find the probability that $(A' \vee B) \wedge (A \vee C')$ is true.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) NOTA

4) The rational numbers have the same cardinality as which of the following?

I) Integers
II) Positive integers
III) Real numbers
IV) Complex numbers whose real and imaginary parts are both rational

A) I and II ONLY B) I and IV ONLY C) II and IV ONLY
D) I, II, III, and IV E) NOTA

5) Simplify the following statement, where A and B are sets: $(A \cap B) \cup (A' \cup B)'$.

A) A B) A' C) B D) B' E) NOTA

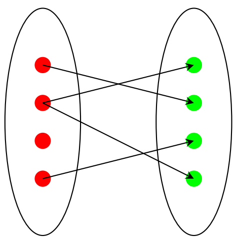
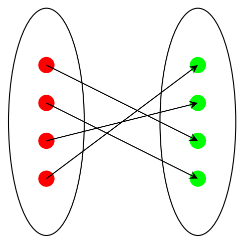
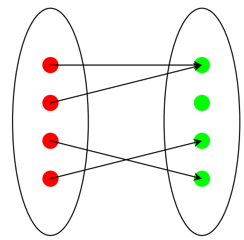
6) Sets A and B have cardinalities 20 and 22 respectively. If $|A \cap B| = 9$, find $|A \cup B|$.

A) 11 B) 24 C) 33 D) 42 E) NOTA

7) Which of the following is equal to $\frac{1}{14}$ in binary?

A) $0.\overline{0001}_2$ B) $0.0\overline{001}_2$ C) $0.00\overline{01}_2$ D) $0.00\overline{011}_2$ E) NOTA

8) Consider the following relations. I of them are injective functions, S of them are surjective functions, and B of them are bijective functions. Find $4B + 2I + S$.

A) 5 B) 6 C) 9 D) 10 E) NOTA

9) A binary tree has 2022 nodes. Find the positive difference between its minimum and maximum height.

A) 2011 B) 2012 C) 2020 D) 2021 E) NOTA

10) Evaluate the following expression, which is written in Reverse Polish (prefix) notation: $1\ 3\ +\ 3\ 7\ +\ \times$.

A) 17 B) 25 C) 31 D) 40 E) NOTA

11) The affine cipher works as follows. Convert each letter of the alphabet to its zero-indexed numerical position, where A = 0, B = 1, C = 2, and so on. For each of these values, multiply it by a constant $0 < a < 26$ such that a and 26 are coprime. Add a constant $0 \leq b < 26$ and take the result modulo 26, converting back into a letter at the end. Using the values of a and b, we have an encryption function $E(x) = ax + b \bmod 26$. Determine what message has been encoded by the affine encryption $E(x) = 3x+7 \bmod 26$ to produce `AFNLNCXFNTN` and act accordingly.

A) A man B) A plan C) A canal D) Panama E) NOTA

12) Find the sum of the digits of the smallest value of n greater than 2022 that satisfies the relations $x \equiv 1 \bmod 4$, $x \equiv 4 \bmod 5$, and $x \equiv 6 \bmod 7$.

A) 11 B) 12 C) 13 D) 14 E) NOTA

For questions $13 - 14$**, you may use the following information.** To find the continued fraction representation of a number, you must proceed through a recursive process. First, find the integer and fractional parts of your number. The integer part is written down as the next term in your continued fraction representation. The reciprocal of the fractional part is put through the first step of the recursive process again. This process will continue until the fractional part is zero (in which case the continued fraction representation terminates) or until the steps start to repeat.

A terminating continued fraction representation is written as $[I; s_1, s_2, \ldots, s_n]$, where I is the integer part of the number and all s_i are positive integers. A repeating continued fraction representation is written as $[I; s_1, s_2, \ldots, s_k, \overline{s_{k+1}, s_{k+2}, \ldots, s_n}]$, where the portion under the bar repeats infinitely.

13) The continued fraction expansion of $\frac{55}{89}$ can be written as $[0; s_1, s_2, \ldots, s_n]$. Find $\sum_{i=1}^{n} s_i$.

A) 8 B) 9 C) 10 D) 11 E) NOTA

14) The continued fraction expansion of $\sqrt{41}$ can be written as $[6; \overline{s_1, s_2, \ldots, s_n}]$. Find $\sum_{i=1}^{n} s_i$.

A) 10 B) 12 C) 14 D) 16 E) NOTA

15) Find the last three digits of 3^{210}.

A) 049 B) 489 C) 729 D) 969 E) NOTA

16) Find the sum of all positive integers x less than 50 such that $x^{86} \equiv 6 \bmod 29$.

A) 66 B) 74 C) 111 D) 116 E) NOTA

17) Find the number of positive integers n such that $\phi(n) = 64$, where $\phi(n)$ is the totient function.

A) 6 B) 7 C) 8 D) 9 E) NOTA

18) Find the last digit of $1+2^{1+2^{1+2^{1+2^{1+2}}}}$.

A) 3 B) 5 C) 7 D) 9 E) NOTA

19) In a cross-sports playoff series, the New England Patriots play the Charlotte Bobcats. The winner of the series is the first team to win four individual games (which must end in either a win or a loss - no ties). Given that there was no point during the series when the New England Patriots had fewer wins than the Charlotte Bobcats, find the number of distinct sequences of wins and losses that could have occurred.

A) 14 B) 21 C) 28 D) 35 E) NOTA

20) $f(x)$ is a polynomial that gives a remainder of 7 when divided by $x-3$ and a remainder of -3 when divided by $x+2$. Find the remainder when $f(x)$ is divided by x^2-x-6.

A) -21 B) $-2x+3$ C) $x+2$ D) $2x+1$ E) NOTA

21) Which of the following statements is false?

A) If the moon is made of green cheese, then a cow will jump over it.

B) If this test is written in LaTeX, then this is answer choice B).

C) If the square root of 4 is -2, then rain makes the streets wet.

D) If Washington, D.C. is the capital of the United States, then it is in the North American Central Time Zone.

E) NOTA

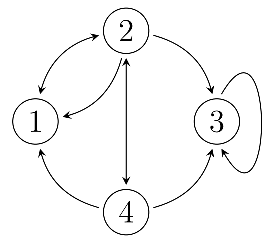
22) Let $N=117^4+64$. N has exactly three prime factors. Find the sum of the digits of the largest prime factor of N. *Hint: You may use the fact that* $117^2=13689$.

A) 14 B) 15 C) 16 D) 17 E) NOTA

23) $\{x_n\}_{n\in\mathbb{N}_0}$ is a sequence of integers with $x_0=1337$, $x_1=2022$, and $x_n=4x_{n-1}+5x_{n-2}$ for all $n\geq 2$. If $\lim\limits_{n\to\infty}\frac{x_n}{K^n}$ exists and is nonzero, find K.

A) 2 B) 3 C) 4 D) 5 E) NOTA

For questions $24-25$, use the following labeled directed graph.



24) Which of these is the adjacency matrix for the graph?

A) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

B) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

C) $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

D) $\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

E) NOTA

25) The Laplacian matrix of a graph is equal to the graph's degree matrix (which has the degree of each node on its main diagonal) minus its adjacency matrix. Find the determinant of the Laplacian matrix for the graph.

A) -48 B) 0 C) 52 D) 249 E) NOTA

26) A solution to which of the following will find the shortest possible route that visits each node exactly once and return to the original node, given a graph of nodes and the distances between them?

A) Eulerian circuit

B) Hamiltonian cycle

C) Knight's tour

D) Minimum spanning tree

E) NOTA

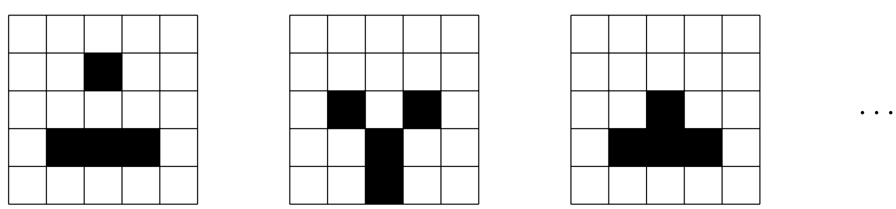
27) Compute the average time complexity of the following pseudocode, which has as its inputs a sorted list of integers `arr`, the number of elements in the array `n`, and the number being searched for `target`. Assume `target` is in `arr`.

1. Let `min = 0` and `max = n-1`
2. Find `guess`, the average of `min` and `max` (rounded down).
3. If `arr[guess] == target`, return `guess`.
4. If `arr[guess] < target`, let `min = guess + 1`.
5. Otherwise, `arr[guess] > target`, so let `max = guess - 1`
6. Go back to step 2.

A) $O(1)$ B) $O(\log\log n)$ C) $O(\log n)$ D) $O(n)$ E) NOTA

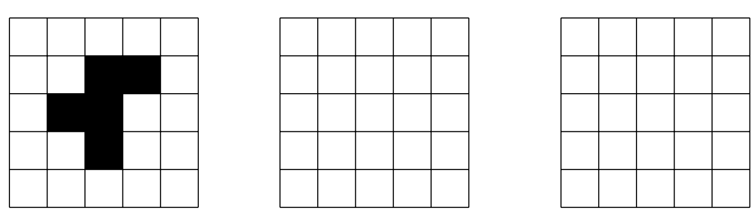
28) Conway's Game of Life takes place on an infinite, two dimensional orthogonal grid of square cells. Each cell can be represented as being in on of two possible states: alive (black) or dead (white). In one generation, each cell interacts with its eight adjacent neighbors according to the following rules, which determine if the cell is alive or dead in the next generation. An example is shown below the rules.

- Any live cell with two or three live neighbors survives.
- Any dead cell with exactly three live neighbors becomes a live cell.
- All other live cells die in the next generation, and all other dead cells stay dead.



Generation 0 Generation 1 Generation 2 . . .

Generation 0 of one instance of Conway's Game of Life is shown below, with an F-pentomino. Determine the number of live cells in Generation 2.



Generation 0 Generation 1 Generation 2

A) 6 B) 7 C) 8 D) 9 E) NOTA

29) Wolfram's Rule automata also take place on an infinitely wide two-dimensional grid. One cell in the top (0th) row is colored black. Each cell in a subsequent row is colored white (0) or black (1) depending on the colors of the three cells that are adjacent to it in the previous row. The digits representing the colors of these cells are concatenated and arranged in the top row of the following chart.

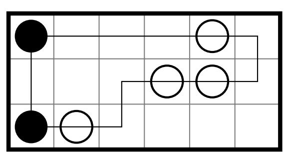
111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

The bottom row of the chart shows the state of the cell in the next generation based on its upper neighbors. The Rule Number of a particular automaton can be found by interpreting the bottom row as an eight-digit binary number (this one is Rule 30). Find the number of black cells in the 5th row of the Rule 30 automaton.

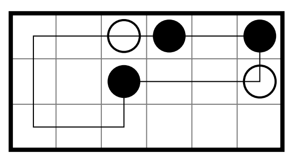
A) 7 B) 8 C) 9 D) 10 E) NOTA

30) A Masyu is a type of logic puzzle that occurs in a rectangular grid of squares. The goal of the puzzle is to draw a loop that properly passes through all circled cells. The loop must "enter" each cell it passes through from the center of one of its four sides and "exit" from a different side; all turns are therefore 90 degrees. The following rules must be abided by.

- White circles must be traveled straight through, but the loop must turn in the previous and/or next cell in its path.
- Black cells must be turned upon, but the loop must travel straight through the next and previous cells in its path.

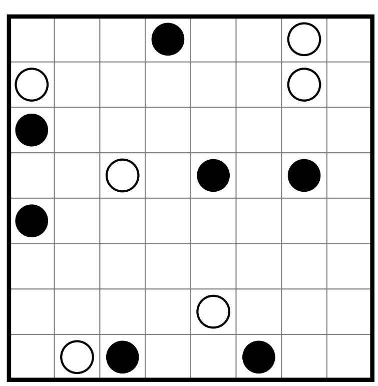


✓ All circles good ✓



× All circles bad ×

A Masyu solution can be viewed as a type of constrained Hamiltonian cycle whose nodes are the circles in the grid. Determine the number of **grid squares** the loop does *not* pass through in the unique solution to the following Masyu.



A) 6 B) 7 C) 8 D) 9 E) NOTA