2022 Mu Alpha Theta National Convention **Washington DC**

Open Number Theory Answers and Solutions

ANSWERS

1) C
2) B
3) A
4) C
5) B
6) B
7) B
8) A
9) C
10) B
11) A
12) C
13) D
14) D
15) C
16) A
17) B
18) E
19) D
20) C
21) C
22) B
23) A
24) D
25) D
26) D
27) A
28) C
29) A
30) E

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1. Let n=2^km with m odd. The sum of the factors of m has the same parity as the sum of the factors of n. Additionally, the number of factors of any number is odd if and only if it is a square. Therefore, the numbers we are looking for are square numbers multiplied by some power of 2. This simplifies down to squares and squares multiplied by 2. There are 31 square numbers below 1000 (31^2=961), and 22 squares multiplied by 2 under 1000 (22^2*2=968), and 31+22=53. **C**
2. 2021=43*47, so the factors are 1, 43, 47, and 2021. **B**
3. 2021! is divisible by 4 (1010+505+252+126+63+31+15+7+3+1)/2=1010 times and is divisible by 3 (673+224+74+24+8+2)=1005 times. Therefore, 2021! is divisible by 12 1005 times, and thus ends in 1005 zeros. **A**
4. The 3 Gaussian primes are 1+i, 2+i, and 3, which give a sum of 6+2i, leading to a final answer of 8. **C**
5. 4^7-1=(2^7+1)(2^7-1)=127*129=3*43*127. The sum of the digits of 127 is 10. **B**
6. The Fibonacci numbers mod 13 cycle with period 28 (1,1,2,3,5,8,0,8,8,3,11,1,12,0,12,12,11,10,8,5,0,5,5,10,2,12,1,0). This pattern shows that every 7th Fibonacci number is divisible by 13. Therefore, we can calculate that exactly ⌊500000/7⌋=71428 Fibonacci numbers are divisible by 13. **B**
7. This is equivalent to asking for which positive integer x does the equality $(x^2+2x+3)(3x^2+2x+1)=3x^4+9x^3+3x^2+8x+3$ hold. When simplified, this equation becomes $11x^2=x^3$, which has solutions of x=0 and x=11. **B**
8. The double factorial means 2020!!=2020*2018*2016*…*4*2. However, this can be expressed as 2^1010*1010!. Since 1009 is prime, that is the final answer. **A**
9. N and P are both infinite and contained in Z, so they have the same cardinality. Q can be represented by pairs of integers (a/b), so it has the same cardinality as Z^2, which has the same cardinality as Z. The complex numbers C are composed of two Real numbers (a+bi), and is therefore equivalent to R^2, which has a larger cardinality than Z. **C**
10. The last 2 digits of 7^n has a cycle of length 4 (it goes 07, 49, 43, 01, 07, …). Since 5296 is divisible by 4, 7^{5296} ends in 01, so 7^{5296}+3 ends in 04, which sums to 4. **B**
11. The Fibonacci numbers will always take the most steps in the Euclidean Algorithm. This stems from the fact that the Fibonacci numbers are directly related to the continued fraction of the golden ratio, which is entirely composed of 1s. Therefore, the answer is 610 and 377. **A**
12. A perfect number is equal to the sum of its proper factors. 28=1+2+4+7+14. **C**
13. 31 is congruent to 8 mod 23, so we can instead calculate 8^{16} mod 23. 8^2 is congruent to 18 mod 23, 18^2 is congruent to 2 mod 23, and 2^4=16. **D**
14. By Fermat's Little Theorem, 31^{23} is congruent to 31 mod 23. Therefore, 31^{25} is congruent to 31^3 mod 23, which is congruent to 8^3=512, which is congruent to 6. **D**
15. 6+8+10=24=6*8/2, and 5+12+13=30=5*12/2. All primitive right triangles with longer sides have more area than perimeter, and thus no dilation will have them be equal. Therefore, the answer is 2. **C**
16. $\sqrt{37}=6+(\sqrt{37}-6)=6+1/(6+\sqrt{37})=6+1/(12+1/(6+\sqrt{37}))$. Therefore, √37 has a period of 1, which is odd. **A**

17. 576 has 21 positive factors, and the factors not equal to 24 can be paired up into pairs that multiply to 576. Therefore, the product of the positive factors is equal to $576^{10}*24=24^{21}$. **B**
18. If we consider the units digits of the answers when substituted into x^6-6x, the only one that could possibly work is 27. However, 27^6-6*27=387420327, so that's not the answer. **E**
19. $10!/7=(6!)^2$. 10+6+7=23. **D**
20. 24867 in BCD is 0010 0100 1000 0110 0111, which has 8 ones. 24867 in binary is equal to 110000100100011, which has 6 ones. Together these sum to 14. **C**
21. ${}_{49}C_0$ and ${}_{49}C_{49}$ are both equal to 1, but since 49! has 8 factors of 7 but any k!(49-k)! will only have 7 for any 0<k<49, A has 48 elements divisible by 7. By Pascal's Triangle, B will have 47 elements divisible by 7. Our final sum is then 48+47=95. **C**
22. $(2n)!!=(2n)*(2n-2)*(2n-4)*\ldots*4*2=2^n*(n)*(n-1)*\ldots*2*1=n!*2^n$. Therefore, $\sum_{n=0}^{\infty}\frac{1}{(2n)!!}=\sum_{n=0}^{\infty}\frac{1}{n!2^n}=\sum_{n=0}^{\infty}\frac{0.5^n}{n!}=e^{0.5}=\sqrt{e}$. **B**
23. The smallest positive solution of this system of congruences, by the Chinese Remainder Theorem, is 138. Every other solution will be an integer multiple of 5*7*11=385 from 138. We have 138+15*385=5913. **A**
24. By Wilson's Theorem, all prime numbers satisfy this equation, and all composite numbers do not. 1 also satisfies this equation, which brings the total to 26. **D**
25. 2^{14}-1 and 2^{16}-1 are not prime by difference of squares, and 2^{15}-1 is not prime due to difference of cubes. 2^{17}-1 is prime and is in fact a Mersenne prime. **D**
26. $\phi(x)$ is the number of positive integers below x that are relatively prime to x. $\phi(300)=80$, $\phi(15)=8$, and $\phi(20)=8$ as well. Therefore $\phi(300)/\phi(15)=10=5\phi(20)/4$. **D**
27. The smallest positive integer to have exactly 64 factors is $(2^3)(3^3)(5)(7)=7560$. This has a sum of digits of 17. **A**
28. The sum of all the numbers from 1 to 16 is 136, so the sum of 1 individual row or column is 136/4=34. Therefore, the top left square has 34-11-14-1=8, and the bottom right square has 34-12-6-1=15. Additionally, we can say the second square in the second row has value 15-A, and the third square in the third row has value 13-B (from the column sum). We can then calculate the sum of the main diagonal as 8+(15-A)+(13-B)+15=51-A-B=34. This implies A+B=51-34=17. **C**
29. By stars and bars, we have 8 stars (players) and 79 bars (80 characters-1). This gives us ${}_{87}C_8$=(87*86*85*84*83*82*81*80)/8!. The numerator is divisible by 3 6 times, and the denominator is only divisible by 3 2 times. Therefore, ${}_{87}C_8$ is divisible by 9, and the answer is 0. **A**
30. After expanding the polynomials and canceling terms, the equation simplifies to $x^3+y^3=z^3$. By Fermat's Last Theorem, this equation has no solutions in the integers. **E**