Open Probability and Combinatorics Answers and Solutions

ANSWERS :

1. A
2. B
3. C
4. C
5. C
6. A
7. D
8. B
9. C
10. D
11. C
12. B
13. C
14. D
15. B
16. B
17. D
18. D
19. B
20. C
21. A
22. A
23. C
24. D
25. A
26. B
27. C
28. A
29. C
30. A

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1. The color of the first sock that Mark draws doesn't matter. The only thing that matters is that the color of the second sock he draws has to match the first sock. Since he already drew a sock, there are 17 total socks, and 5 of the color he wants. Therefore, the probability is 5/17. **A**
2. As previously stated, the probability of getting it in 2 socks is 5/17. If he doesn't get it in 2 socks, there are 2 colors that he could draw and have a matching pair on his third draw. This results in a probability of 12/17*10/16=15/34. If he still doesn't have a matching pair, he has a sock of every color, which means he is guaranteed a pair on his next draw. Thus, we have an expected value of 2(5/17)+3(15/34)+4(1-5/17-15/34)= 20/34+45/34+36/34=101/34. **B**
3. By overlaying Pascal's Triangle on the Cartesian Plane, we can see that this is equivalent to ${}_{11}C_4$=330. **C**
4. Since all paths must travel through (1,3), this splits the problem into 2 smaller problems. There are 4 possible ways to get to (1,3) from (0,0), and 35 possible ways to get to (4,7) from (1,3). Therefore, there are a total of 140 paths. **C**
5. Assume the horizontal sides will be taken from the side of length 4, and the vertical sides will be taken from the side of length 6. There are 10 possible horizontal sides (4 of length 1, 3 of length 2, 2 of length 3, and 1 of length 4). Similarly, there are 21 possible vertical sides. Therefore, there are 210 possible rectangles. **C**
6. If Alex chooses a 2-handed weapon, there are 4 options. If he chooses 2 1-handed weapons, there are 6(5)/2=15 options for 2 different weapons and 6 options for 2 of the same weapons. If he chooses a weapon and a shield, there are 3(6)=18 options. This gives a total of 4+15+6+18=43 total options. **A**
7. For Alex to miss, he would have to roll 9 or below on both of his d20s. Since rolling 9 or below is a 45% chance, 2 of those is 0.45*0.45=.2025. Therefore, Alex's expected damage is equal to 9(1-0.2025)=9(.7975)=7.1775, which rounded to the nearest whole number is 7. **D**
8. 1 and 12 map to 1, 2 and 11 map to 4, 3 and 10 map to 9, 4 and 9 map to 3, 5 and 8 map to 12, and 6 and 7 map to 10. Therefore, 1 repeats after 1 iteration, 3 and 12 repeat after 2 iterations, 4, 5, 8, 9, and 10 repeat after 3 iterations, and the rest repeat after 4 iterations. Therefore, the expected number of times is (1*1+2*2+5*3+4*4)/12=36/12=3. **B**
9. The chances of Steven losing each game individually are 1/2^1, 1/2^2, …, 1/2^10. Therefore, the chances of Steven losing all 10 games are 1/2^(1+2+…+10)=1/2^55. **C**
10. Every number that satisfies this condition is a non-empty substring of "9876543210", of which there are 1023. **D**

11. To be divisible by 4, a number's last 2 digits have to be divisible by 4, which forces the last digit to be even. If the last digit is 0, the second last digit must be 2, 4, 6, or 8, which give 128, 32, 8, and 2 possible numbers respectively. If the last digit is 2, the second last digit must be 1, 3, 5, or 7, which give 64, 16, 4, and 1 possible numbers respectively. If the last digit is a 4, the second last digit must be a 6 or 8, which give 8 and 2 possible numbers respectively. If the last digit is a 6, the second last digit must be a 7 or 9, which gives 8 and 1 possible numbers respectively. The last digit can't be an 8, because 98 isn't divisible by 4. Summing all these numbers up gives us 270 numbers satisfying the criterion. **C**
12. This is equivalent to 2-coloring a complete graph of n vertices. 5 people is not enough, as you can color the edges of a pentagon red and the diagonals blue, and there wouldn't be a monochromatic triangle. However, it can be shown that a group of 6 people guarantees a monochromatic triangle. Therefore, the answer is 6. **B**
13. This is called the number of derangements, and it is equal to the closest integer to n!/e. 5!/e=120/e≈44.15, so the answer is 44. **C**
14. There are 900 possible rolls of 2 30-sided dice. There is 1 where the highest roll is 1, 3 where the highest roll is 2, 5 where the highest roll is 3, etc. In general, there are (2n-1) situations where the highest roll is n. Therefore, we want to solve the sum $\frac{1}{900}\sum_{i=1}^{30} i(2i-1) = 2i^2 - i$. The sum of the first j squares is $\frac{j(j+1)(2j+1)}{6}$, and the sum of the first k integers is $\frac{k(k+1)}{2}$. Therefore, the sum is equal to $\frac{1}{900}\left(\frac{30(31)(61)}{3} - \frac{30(31)}{2}\right) = \frac{1}{900}(18910 - 465) = \frac{18445}{900} \approx 20.49$, which rounds down to 20. **D**
15. Since we can rotate the table, assume Jake is sitting at the top. Since Arthur doesn't want to sit next to him, he has 4 available seats to choose from. Once he chooses his seat, the other 5 don't have any preferences among the 5 remaining seats, giving a result of 4*5!=480. **B**
16. There are various ways to make a constant with this expansion. 4^4 can be created in 1 way, $4^2(2x)(1/x)$ can be created in 4!/2!=12 ways, 4(3x^2)$(1/x)^2$ can be created in 12 ways, and $(2x)^2(1/x)^2$ can be created in 6 ways. This gives us the sum 256+32*12+12*12+4*6=808. **B**
17. This is equivalent to the 10^{th} Catalan number, which has the formula $1/(n+1)*{}_{2n}C_n$. Plugging this in gives $1/11*{}_{20}C_{10}$=132. **D**
18. The formula is ${}_nC_2$, which gives ${}_{15}C_2$=105. **D**
19. Clearly ${}_3C_0={}_4C_0$. Additionally, ${}_4C_0+{}_4C_1={}_5C_1$ by Pascal's Triangle. This same identity lets us cascade the sum into ${}_{13}C_9$=715. **B**

20. If 1 is the lowest number, all the other numbers have to be included, giving 1 set. If 2 is the lowest number, then 4, 6, and 8 have to be included as well. 5 and 7 are independent of the other numbers, but if 3 can't be in it without 9. This gives 2*2*3=12 sets. If 3 is the lowest number, then 6 and 9 have to be included. Similarly to the last case, 5 and 7 are independent, but 4 can't be in without 8. This gives another 12 cases. If 4 is the lowest number, then 8 has to be in the set. However, this time all the other numbers are independent, giving 16 sets. Above this, there are no dependent elements, so 5 gives 16 sets, 6 gives 8, 7 gives 4, 8 gives 2, and 9 gives 1. Summing all these up gives 72 sets. **C**
21. The number of ways all 5 members are male is ${}_8C_5$=56. The number of ways 4 members are male and 1 is female is ${}_8C_4$*12=70*12=840. The number of ways 3 members are male and 2 are female is ${}_8C_3$*${}_{12}C_2$=56*66=3696. This gives a total of 4592. **A**
22. For this to happen, first positions 6 and 7 need to be heads, and position 5 needs to be tails (otherwise 5 and 6 would be a streak of 2 heads). Then, there can't be a streak of 2 heads in positions 1-4. This occurs in 8 of the options (TTTT,TTTH,TTHT,THTT,HTTT,THTH,HTHT,HTTH). Therefore, since there are 128 total ways to flip the coins, the answer is 8/128=1/16. **A**
23. Let the probability that Lucas gets his free throw be p. There is then a p+(1-p)(1-0.6)p+(1-p)^2(1-0.6)^2p+… chance that Lucas wins. However, we know that this chance is 0.5. Therefore, we can use the sum of an infinite geometric series formula. 0.5=p/[1-(1-p)(0.4)]. Solving for p we get p=0.6/1.6=3/8. **C**
24. To end up at 1, Gus needs to have walked right 3 times and left 2 times. The number of orderings of these steps is ${}_5C_3$=10. Since there are 32 possible sets of steps he could have taken, the answer is 10/32=5/16. **D**
25. The sum of positive pairwise differences is always twice the largest positive difference. Since it is always even, it can never be 9, and any event is mutually exclusive with an event with 0 probability. **A**
26. There is a 1/9 chance the first digit is less than 2 (as it can't be 0). There is a 2/10=1/5 chance that any of the other digits are less than 2. Using linearity of expectation, the expected number is 1/9+5*1/5=10/9. **B**
27. Your expected profit after 1 game is 5(p)-5(1-p)=5(2p-1). Therefore, your expected profit after 30 games is 30(5)(2p-1), as the games are independent. Now this is a simple equation. 300p-150=45, 300p=195, p=0.65. **C**
28. By stars and bars, the answer is ${}_{(11-1)}C_{(5-1)}={}_{10}C_4$=210. **A**
29. By stars and bars, the answer is ${}_{(11+5-1)}C_{(5-1)}={}_{15}C_4$=1365. **C**
30. The digits of pi are uniformly distributed (or close enough to be negligible) over the numbers 0-9, so the probability that any given digit is an 8 is 0.1 **A**