THETA TEAM ANSWERS and SOLUTIONS
2022 MAO NATIONAL CONVENTION

ANSWERS :

0. 7175
1. 1044
2. 20
3. 201
4. 100
5. 2112
6. 108
7. -86
8. 27
9. 113
10. 9
11. 2652
12. 58
13. 2300
14. 125

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SOLUTIONS :

Question 0

$A=3$	$2019=3\times673$
$B=101$	$2020=2^2\times5\times101$
$C=2021$	
$D=5050$	$\frac{100(101)}{2}=5050$

$A+B+C+D=7175$

Question 1

A = 1260: The medals are indistinguishable, so we have $\frac{9!}{4!3!2!}=1260$

B = 4320: Consider the three vowels as one word, leaving 6 "letters." There are 3! ways to arrange the vowels and 6! ways to arrange the 6 letters. (3!)(6!) = 4320.

C = 720: Place the D and R, leaving 6 unique letters: 6! = 720.

D = 100: We are looking for numbers 101-999. We have 5 available for the first digit, 5 for the second digit, and 4 for the final digit: (5)(5)(4) = 100.

(1260 – 100)/10 = 116; (4320 – 720)/100=36. $116=2^2\times29$, $36=2^2\times3^2$. $LCM=2^2\times3^2\times29=1044$.

Question 2

A = 18, B = 36: The enclosed region is a semicircle atop a triangle, like a cross-section of a single-scoop of ice cream on a cone: $\frac{1}{2}\pi(6)^2+\frac{1}{2}(12)(6)=18\pi+36=A\pi+B$.

C = 27, D= 10: The points (–2, 4) and (–1, 1) conveniently lie on the line $y=-3x-2$, and the resulting solid will be two cones whose base radii are the altitude from (–4, 1) to the line. The radii are $\frac{|3(-4)+(1)+2|}{\sqrt{3^2+1^2}}=\frac{9}{\sqrt{10}}$. Since the cones share the same base, we can find the distance between (–2, 4) and (–1, 1) and not have to find the height of each cone: $\sqrt{1^2+3^2}=\sqrt{10}$. $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{9}{\sqrt{10}}\right)^2\left(\sqrt{10}\right)=\frac{27\pi\sqrt{10}}{10}=\frac{C\sqrt{D}\pi}{D}$.

$$\frac{AD}{B-C}=\frac{(18)(10)}{36-27}=\frac{180}{9}=20$$

Question 3

A = 8: (1, 210), (2, 105), (3, 70), (5, 42), (6, 35), (7, 30), (10, 21), (14, 15)

B = 144: $\frac{211+107+73+47+41+37+31+29}{4}=144$

C = 21: 4 "large" rectangles, 7 "1x1", 8 "2x1", and 2 "3x1"

D = 28: $(w+2)\left(\frac{l}{2}\right)=lw \rightarrow lw+2l=2lw \rightarrow w=2$, since $l \neq 0$; $2l+2w=(1.4)\left[2(w+2)+2\left(\frac{l}{2}\right)\right]\rightarrow$

$l+w=(1.4)\left(w+2+\frac{l}{2}\right)\rightarrow l+2=(1.4)\left(4+\frac{l}{2}\right)\rightarrow .3l=3.6\rightarrow l=12;\ D=2(2+12)=28$

$A+B+C+D=8+144+21+28=201$

Question 4

A = 89: $\begin{cases} a^2+b^2=169 \\ ab=40 \end{cases} \rightarrow \begin{cases} a^2+b^2=169 \\ -2ab=-80 \end{cases} \rightarrow a^2-2ab+b^2=89 \rightarrow (a-b)^2=89,$

B = 189: h is the opposite of the product of the roots. Let the roots be $r, r+2, s$. Sum of roots:

$2r+2+s=13\ 2r+s=11 \rightarrow s=11-2r$. Sum of roots taken two at a time:

$(r^2+2r)+(rs)+(rs+2s)=15=r^2+2r+r(11-2r)+2(11-2r)\rightarrow 3r^2-20r-7=0=$

$(3r+1)(r-7)\rightarrow r=-\frac{1}{3}, 7.$ Now we have roots $-\frac{1}{3}, \frac{5}{3}, \frac{35}{3} \rightarrow h=\frac{175}{27}$ or $7, 9, -3 \rightarrow h=189.$

$189-89=100$

Question 5

A = 5: $(54)(2.5)+54T=81T \rightarrow 135=27T \rightarrow T=5$

B = 8800: $10R=5\left(\frac{1}{2}-R\right)\rightarrow 2R=\frac{1}{2}-R \rightarrow R=\frac{1}{6}$ hr $\rightarrow 10\left(\frac{1}{6}\right)(5280)=8800$ ft

C = 6: $0.02x-0(100)=0.03(x-100)\rightarrow 2x=3x-300\rightarrow x=300\rightarrow 300(0.02)=6$

D = 5: $x(12-2x)(18-2x)=160 \rightarrow x(6-x)(9-x)=40 \rightarrow x^3-15x^2+54x-40=0 \rightarrow$

$(x-1)(x-10)(x-4)=0$. Only $x=1$ and $x=4$ are in the domain. Sum is 5.

$\frac{(8800)(6)}{(5)(5)}=2112$

Question 6

$A = -2$: Let the four terms be a, ar, ar^2, ar^3. Setting up the given information as a quotient, we have

$$\frac{ar^2-a}{ar-ar^3}=\frac{9}{18}=\frac{a(r+1)(r-1)}{-ar(r+1)(r-1)}=\frac{1}{2}=\frac{1}{-r}\rightarrow r=-2$$

$B = 11$: $11+(11+d)+(11+2d)+(11+3d)=56=44+6d\rightarrow 22+3d=28\rightarrow d=2$

$$[11+2(n-1)]+[11+2(n-2)]+[11+2(n-3)]+[11+2(n-4)]=112=44+2(4n-10)\rightarrow n=11$$

$C = -13$: GP: $a+ar+ar^2=56=a(1+r+r^2)$. AP: $2(ar-7)=(a-1)+(ar^2-21)\rightarrow$

$2ar-14=ar^2+a-22$. Add ar to each side so we can substitute from the GP:

$(ar)+2ar-14=(ar)+ar^2+a-22\rightarrow 3ar-14=56-22\rightarrow ar=16\rightarrow a=\frac{16}{r}$.

Find r as in part A: $\frac{a(1+r+r^2)}{a(r^2-2r+1)}=\frac{56}{8}=\frac{1+r+r^2}{r^2-2r+1}=\frac{7}{1}\rightarrow 1+r+r^2=7r^2-14r+7\rightarrow$

$6r^2-15r+6=0\rightarrow(6r-3)(r-2)=0\rightarrow r=\frac{1}{2}, 2$. The AP terms are 31, 9, −13 or 7, 9, 11.

The smallest is −13.

Question 7

A: $f(x)=2x^2-24x-90$: Let the roots be r and $r-18$. Sum $=2r-18=12\rightarrow r=15$. The roots are 15 and −3. $\frac{c}{2}=(15)(-3)\rightarrow c=-90\rightarrow f(x)=2x^2-24x-90$.

B: $x=y^2-3y+1$: $\begin{cases}-1=a+b+c\\ 11=4a-2b+c\\ 5=a-b+c\end{cases}$. Subtracting the first equation from the third: $6=-2b\rightarrow b=-3$.

Substituting this value and then subtracting those same equations: $\begin{cases}2=a+c\\ 5=4a+c\end{cases}\rightarrow 3=3a\rightarrow$

$a=1, c=1$. $x=y^2-3y+1$.

$2+(-90)+1+1=-86$

Question 8

$A = 4$: The only integer solutions are (1, 2), (2, 1), (−1, −2), (−2, −1).

$B = 4$: $y=\frac{2x-2}{x-2}=\frac{2}{x-2}+2$, which is the graph from the previous problem moved up 2 and over 2. The number of integer solutions doesn't change.

$C = 5$: (100, 1), (79, 21), (58, 41), (37, 61), (16, 81)

$D = 14$: The third angle in the smaller triangle has measure $51°$, as does the angle adjacent to it. The adjacent angle to the $58°$ angle is $122°$. This leaves $7°$ for the other half angle, so $m\angle ABC=14°$.

$4+4+5+14=27$

Question 9

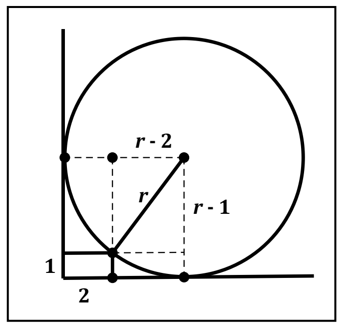
A = 13: $\frac{n(n-3)}{2}=5n\to\frac{n-3}{2}=5\to n=13$

B = 20: Let domain of f is $(-\infty, 6)$ and the domain of g is $(-\infty, \infty)$. $f\circ g=\ln\left(6-\left|x^2-10x+15\right|\right)$. This domain must satisfy $6-\left|x^2-10x+15\right|>0\to x^2-10x+15<6\cap x^2-10x+15>-6\to$ $(x-9)(x-1)<0\cap(x-7)(x-3)>0\to(1, 9)\cap\left[(-\infty, 3)\cup(7, \infty)\right]\to(3, 1)\cup(7, 9)$.

$3+1+7+9=20$

C = 5: $(r-2)^2+(r-1)^2=r^2\to 2r^2-6r+5=r^2\to(r-5)(r-1)=0\to r=5$

$13+20(5)=113$



Question 10

(A, B) = (3, −3): $-6+24i=(x-iy)(3+5i)=(3x+5y)+(5x-3y)i\to\begin{cases}3x+5y=-6\\5x-3y=24\end{cases}\to(3, -3)$

C = 2: $\frac{1+i}{1-i}-\frac{1-i}{1+i}\to\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}=\frac{2i-(-2i)}{2}=2i\to|2i|=2$

D = 1: $\frac{(\log_8 3)(\log_7 125)}{(\log_{49} 25)(\log_4 9)}\to\frac{\left(\log_{2^3} 3\right)\left(\log_7 5^3\right)}{\left(\log_{7^2} 5^2\right)\left(\log_{2^2} 3^2\right)}\to\frac{\left(\frac{1}{3}\log_2 3\right)\left(3\log_7 5\right)}{\left(\log_7 5\right)\left(\log_2 3\right)}=1$

$\begin{vmatrix}3 & -3\\2 & 1\end{vmatrix}=3-(-6)=9$

Question 11

A = 18564: $\binom{18}{12}(9)^6\left(-\frac{1}{3}\right)^{12}=\frac{18!(9)^6}{12!6!(3)^{12}}=\frac{18!(3)^{12}}{12!6!(3)^{12}}=\frac{18!}{12!6!}=18564$

B = 1: $\binom{50}{16}(2)^{34}(a)^{16}=\binom{50}{17}(2)^{33}(a)^{17}\to\frac{50!}{34!16!}(2)^{34}(a)^{16}=\frac{50!}{33!17!}(2)^{33}(a)^{17}\to\frac{2}{34}=\frac{a}{17}\to a=1$

C = 7: $\left(1+x+x^2\right)^3\to(1+y)^3=1+3y+3y^2+y^3\to1+3\left(x+x^2\right)+3\left(x+x^2\right)^2+\left(x+x^2\right)^3$

The x^3 terms will be in the third and fourth terms. The third term becomes $3\left(x^2+2x^3+x^4\right)$ and the first term in the expansion of the fourth term will be x^3. The sum is $7x^3$.

$\frac{18564}{(1)(7)}=2652$

Question 12

$12y^2-4x^2+72y+16x+44=0\to12y^2+72y-4x^2+16x=-44\to3y^2+18y-x^2+4x=-11\to$

$3\left(y^2+6y+9\right)-\left(x^2-4x+4\right)=-11+27-4=12\to\frac{(y+3)^2}{4}-\frac{(x-2)^2}{12}=1\to a=2,\ b=2\sqrt{3},\ c=4$

$(A,B)=(2,-3)$

$(C,D),(E,F)=(2,-1),(2,-5)$

$(G,H),(I,J)=(2,1),(2,-7)$

$K=\frac{c}{a}=2$

$L=(2b)^2=\left(4\sqrt{3}\right)^2=48$

$M=2a=4$

$N=\frac{2b^2}{a}=12$

$O,P:\ y=-3\pm\frac{a^2}{c}=-3\pm\frac{4}{4}=-2,-4$

$Q=\left[\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\right]^{-1}=3$

$2-3+2-1+2-5+2+1+2-7+2+48+4+12-2-4+3=58$

Question 13

A = 10: $a^2+b^2+c^2=200$. We know that, in a right triangle with hypotenuse c, $a^2+b^2=c^2$, so $c^2+c^2=200 \rightarrow c=10$.

B = 23: $P_{\triangle ABX}=AB+BX+AX$, $P_{\triangle XBY}=BX+BY+XY$, $BX=BX$, and $AX=XY$. Since $AB>BY$, $AB-BY$ is the positive difference. Let M be the midpoint of $\overline{AC}$ so that $\overline{BM}\perp\overline{AC}$.

$\left(6\sqrt{3}\right)^2+2^2=\left(BY\right)^2 \rightarrow BY=4\sqrt{7}$. $AB-BY=12-4\sqrt{7}\rightarrow 12+4+7=23$

C = 60: $\triangle ABC$ is equilateral, so the angle measure is 60 degrees.

D = $\frac{10}{7}$: Use similar triangles to find the height x of the triangle: $\frac{x}{2}=\frac{5}{7}\rightarrow x=\frac{10}{7}$

$(23)\left(\frac{10}{7}\right)(70)=2300$

Question 14

$\begin{cases}2x+3y=11\\2x-4y=-24\end{cases}\rightarrow 7y=35\rightarrow y=5,\ x=-2\rightarrow(-2,5)\rightarrow 5=-2m+3\rightarrow m=-1$

$(x-4)^2+((-x+3)+2)^2=5\rightarrow x^2-8x+16+x^2-10x+25=5\rightarrow x^2-9x+18=0\rightarrow(x-6)(x-3)=0\rightarrow$ $(6,-3),(3,0)$

$\sqrt{(6-0)^2+(-3-y)^2}=\sqrt{(3-0)^2+(0-y)^2}\rightarrow 36+9+6y+y^2=9+y^2\rightarrow y=-6\rightarrow(0,-6)$

$(-2-0)^2+(5-(-6))^2=125$