**Answers and Solutions**

**ANSWERS**

1. B
2. A
3. B
4. D
5. C
6. C
7. A
8. A
9. B
10. B
11. D
12. E (0, 0)
13. A
14. C
15. A
16. D
17. E (III, IV)
18. A
19. D
20. D
21. B
22. D
23. C
24. E ($\frac{23}{5}$)
25. C
26. C
27. C
28. A
29. B
30. B

**SOLUTIONS**

1. $\left(a^2+b^2+c^2\right)p^2-2\left(ab+bc+cd\right)p+\left(b^2+c^2+d^2\right)\le 0 \to \left(ap-b\right)^2+\left(bp-c\right)^2+\left(cp-d\right)^2\le 0$. This can only be true if each expression in parentheses has a value of 0. Each of these expressions can be written being equivalent to *p*: $p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$. *b* is the geometric mean between *a* and *c*, and *c* is the geometric mean between *b* and *d*. Therefore, *a*, *b*, *c*, and *d* are in geometric progression, **B**.
2. The line must be vertical and have an *x*-value of -20: **A**.
3. Expanding with the top row: $x\left(-3x^2-6x-2x^2+6x\right)+6\left(2x+4+3x-9\right)-1\left(4x-9x\right)=0\to$ $-5x^3+35x-30=0\to x^3-7x+6=0\to(x-1)(x+3)(x-2)=0$. So $x=0$ is not the value. **B**
4. $8y+7=2y^2+3x\to 2y^2-8y+\left(3x-7\right)=0$. We now need the discriminant to be 0: $64-4(2)(3x-7)=120-24x=0\to x=5$, **D**.
5. We know that $AM>HM$, so $\frac{a+c}{2}>b$ and $\frac{b+d}{2}>c$. Adding these together and simplifying gives us choice **C**.
6. $z=x^y=\left(y^z\right)^y=y^{yz}\to z^x=\left(y^{yz}\right)^x=y^{xyz}$, **C**.
7. The smallest value of *x* will be 2, increasing by 3 until $x=35$. $2+11(3)=35$, a total of 12, **A**.
8. $M!=(7!)(5\bullet4\bullet3\bullet2)(3\bullet2)=(7!)(4\bullet2)(3\bullet3)(5\bullet2)=10!$, **A**.
9. There are six arrangements to consider (in order from greatest to least) from the given information: *cabd*, *cadb*, *cdab*, *acdb*, *acbd*, and *abcd*. The last two are the only ones that satisfy the requirements. 2/6 = 1/3, **B**.
10. This is a stars-and-bars problem, analogous to distributing 12 candies to 3 children where each child must be given at least one piece of candy. First, give each child one piece of candy so that the requirement is met. This leave 9 candies. ${}_{9+3-1}C_{3-1}={}_{11}C_2=55$, **B**.

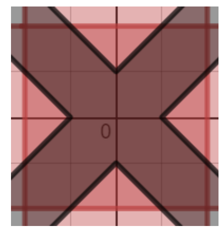
11. $\log_2 x + \log_2 y \ge 6 \to \log_2 xy \ge 6 \to xy \ge 64$. By definition $AM \ge GM$, so we have $\frac{x+y}{2} \ge \sqrt{xy} \to$ $x + y \ge 2\sqrt{xy}\left(=2\sqrt{64}=16\right)$. **D**
12. This is a right triangle and the vertex of the right angle is at the origin. The orthocenter of a right triangle the vertex of the right angle, so this orthocenter is at (0, 0), **E**.
13. The triangle has perpendicular sides of length 1, so the area is 0.5, **A**.
14. $x^2+2x-15>0 \to (x+5)(x-3)>0 \to (-\infty, -5)\cup(3, \infty)$. The midpoint between –5 and 3 is –1, and the distance from –1 to –5 and 3 is 4. The inequality symbol stays the same. **C**
15. $2\left({}_nP_3\right)=3\left({}_{n-1}P_3\right) \to 2n(n-1)(n-2)=3(n-1)(n-2)(n-3) \to 2n^2-4n=3n^2-15n+18 \to$ $n^2-11n+18=0 \to (n-9)(n-2)=0 \to n=2, 9.$ We cannot use $n=2,$ so we are left with $\frac{(9!)^2}{(10)!(8)!}=\frac{9}{10}$, **A**.
16. $x^2+6x-6\sqrt{x^2+6x-2}+3=0 \to x^2+6x-2-6\sqrt{x^2+6x-2}+5=0 \to a^2-6a+5=0 \to$ $(a-5)(a-1)=0 \to x^2+6x-2=25,\ x^2+6x-2=1 \to (x+9)(x-3)=0,\ x^2+6x-3=0 \to$ $x=-9,\ x=3,\ x=-3\pm2\sqrt{3}.$ $a+b+c+d=-9+3-3+2=-7$, **D**.
17. The values that satisfy these two equations are in III and IV, which is not listed. **E**
18. $(3-x)^3=3x^3+3x^2-31x+27 \to 27-27x+9x^2-x^3=3x^3+3x^2-31x+27 \to 4x^3-6x^2-4x=0 \to$ $2x\left(2x^2-3x-2\right)=0 \to 2x(2x+1)(x-2)=0 \to x=0, -\frac{1}{2}, 2,$ but 2 gives a base of 1, which isn't defined. If $x=0$: ${}_9C_7=36$; if $x=-\frac{1}{2}$, ${}_8C_8=1$. **A**
19. $\frac{\left(2^{x+y}\right)\left(2^{x+z}\right)}{2^{y+z}}=\frac{(10)(30)}{20}=2^{2x}=15 \to 2^x=\sqrt{15}$, **D**.
20. The first number is 1/8 of the sum and the second number is 1/3 of the sum, for a total of 11/24 of the sum. That leaves 13/24, so the coefficient of *k* is 24/13, **D**.
21. By definition, this is an equation for a horizontal ellipse with foci at –3 and 1 on the real (*x*-) axis. The absolute values represent the lengths of the focal radii whose sum is 2*a*, so $a=4$. The vertices are the farthest from the center and are located at –5 and 3. Substituting these two values into $|z-4|$ gives us 1 and 9, **B**.
22. Using the distance formula: $\sqrt{(x-6)^2+(y-0)^2}=\sqrt{(x-0)^2+(y-8)^2} \to -12x+36=-16y+64 \to$ $3x-4y=-7$, **D**.
23. We have to look at two cases: $x-2\ge0$ and $x-2<0$. For the first case, we get $x^2-(x+2)+x>0 \to x^2-2>0 \to \left(-\infty, -\sqrt{2}\right)\cup\left(\sqrt{2}, \infty\right)$. For the second case, we get $x^2-(-x-2)+x>0 \to x^2+2x+2>0 \to (x+1)^2+1>0$, which is true for all reals. The solution set for the first case is the only one that works for both cases, so $|ab|=\left|\left(-\sqrt{2}\right)\left(\sqrt{2}\right)\right|=2$, **C**.

24. (Note: This is not $AX = B \rightarrow X = A^{-1}B$ as most textbooks only show.) $XA = B \rightarrow X = BA^{-1}$.
$BA^{-1} = \frac{1}{10}\begin{bmatrix} 6 & 9 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & 24 \\ 19 & -6 \end{bmatrix}$. The sum of these values is $\frac{23}{5}$, **E**. (For the equation in the Note, the answer is $\frac{24}{5}$.)

25. Simplify the left side of the equation as $(a+b)^2 - (a-b)^2 \rightarrow 4ab$. Now we have $4\left(10^{12}\right)(25) = 10^{14}$, **C**.

26. If we plug in each value and add the equations, we get $a_1^{2021} + a_2^{2021} + a_3^{2021} + ... + a_{2021}^{2021} =$ $2020\left(a_1 + a_2 + a_3 + ... + a_{2021}\right) + 2021(2019)$. Since the sum of the roots is 0, we are left with 2021(2019)=4,080,399, **C**.

27. The first two inequalities form a square of area 16. The last inequality creates an "X" over the square with four "cutouts" that are isosceles right triangles, each of area 1. Subtracting the area of the triangles, the bounded area is 12, **C**.




28. $(r-4)(r+1)(r+2) = r^3 - r^2 - 10r - 8 = r\left(r^2 - r - 10\right) - 8$. Substituting the given information, we get $r(0) - 8 = -8$, an integer. **A**

29. Since the equation has real coefficients, the conjugate of the given root will also be a root. Let the three roots be $1 \pm i\sqrt{5}$ and $r$. The sum of the roots will be $2 + r = 6$, so $r = 4$. The sum of the product of the roots taken two at a time will be $2r + 6 \rightarrow 14$. The product of all three roots will be $6r \rightarrow 24$. This leaves us with **B**.

30. Since $a$, $b$, and $c$ are in AP, we know that $b - a = c - b \rightarrow a + c = 2b$. Substituting, we get $2b + b = 1.5 \rightarrow b = \frac{1}{2}$, $a + c = 1$. Since the squares of $a$, $b$, and $c$ are in GP, we know that $\frac{b^2}{a^2} = \frac{c^2}{b^2} \rightarrow$ $b^4 = a^2c^2 \rightarrow b^2 = \pm ac$. Substituting into those same expressions, we get $\frac{1}{16} = a^2(1-a)^2 =$ $a^2(a-1)^2 \rightarrow \frac{1}{4} = \pm a(a-1)$. Since $a < b < c$ and $a + c = 1$, $a - 1$ must be negative, so we choose $\frac{1}{4} = -a(a-1) = -ac$. Now we go to Vieta's formulas. We will let $a$ and $c$ be the roots of a quadratic equation. The sum of the roots is 1 and the product of the roots is $-\frac{1}{4}$, giving us the equation $4x^2 - 4x - 1 = 0$, whose roots are $\frac{1 \pm \sqrt{2}}{2}$. Since $a < c$, $a = \frac{1}{2} - \frac{\sqrt{2}}{2}$, **B**.