1) Let $f(2x-1) = 15x$. What is the value of $f(15)$?

Solution
$\boxed{B}$. At $x = 8$, $2x - 1 = 15$, so $f(15) = 15 \cdot 8 = \boxed{120}$.

2) Let $f(x, y) = x^2 + y^2$. What is the value of $f(\sqrt{5} - 1, \sqrt{5} + 1)$?

Solution
$\boxed{D}$. Plugging in both values gives $6 - 2\sqrt{5} + 6 + 2\sqrt{5} = \boxed{12}$.

3) The function $f(x)$ achieves its unique maximum value at (a, b), for a and b real numbers. At what point does the function $f(3x)$ achieve its unique maximum value?

Solution
$\boxed{B}$. Note that this transformation simply compresses the function $f(x)$ by a factor of 3. The x-coordinate of the max becomes $\frac{a}{3}$ and the value of the max does not change.

4) The function $f(x) = 3x^2 - 12x + 1$ is the function $g(x) = 3x^2$ translated m units down and n units to the right on the Cartesian plane. What is the value of $m + n$?

Solution
$\boxed{D}$. We can complete the square to get $f(x) = 3(x-2)^2 - 11$. We must shift by 11 units up and 2 units to the left, as $g(x-2) + 11 = f(x)$, so $m + n = \boxed{13}$.

5) The quadratic function $f(x) = 5x^2 + nx + 13$ (n a real number) can be factored as a product of two linear factors

$$f(x) = (Ax + B)(Cx + D)$$

for positive integers A, B, C, D. What is the minimum possible value of $A + B + C + D$?

Solution
$\boxed{D}$. Note that because A, B, C, D are restricted to positive integers, $A + C = 5 + 1 = 6$ and $B + D = 13 + 1 = 14$ because 5 and 13 are prime, so $A + B + C + D$ is always equal to $\boxed{20}$.

6) What is minimum value that the function $f(x) = 4^x - 2^{x+3}$ reaches on its domain (all reals)?

Solution
$\boxed{A}$. Note that $4^x = 2^{2x}$ and $2^{x+3} = 8 \cdot 2^x$. This is a quadratic in 2^x, so we will complete the square to get $f(x) = (2^x - 4)^2 - 16$. The minimum occurs when the squared term is minimized, which is at $x = 2$. Thus the minimum value is $\boxed{-16}$.

7) The distance between the Cartesian points $(6, m)$ and $(m+5, 2)$ for some real number m is 5. What is sum of all possible values of m?

Solution
$\boxed{B}$. Using the formula for Cartesian distance, we get that (if d is distance)

$$d^2 = (m-1)^2 + (2-m)^2 \implies 25 = 2m^2 - 6m + 5 \implies 2m^2 - 6m - 20 = 0$$

From this, we get that the sum of the values of m (by factoring and solving or Vieta's) is $\boxed{3}$.

8) The region R in the Cartesian Plane is bounded above by $f(x) = x - 4$ and below by the arc in the fourth quadrant of $x^2 + y^2 = 16$. What is the area of R?

Solution
$\boxed{B}$. After drawing the picture, the region being described is a quarter circle with the boundary right triangle removed. This has area $\frac{1}{4}4^2\pi - \frac{1}{2}4^2 = \boxed{4\pi - 8}$.

9) Let $f(x) = \dfrac{1}{x^2+1}$. For how many real values of a does the horizontal line $y = a$ intersect $f(x)$ exactly once?

Solution
$\boxed{B}$. If we draw a rough sketch of $f(x)$, we see that it is an even function with a peak at $(0, 1)$ and asymptotically approaches 0 as points get further from the center. The only point where a horizontal line can intersect exactly once is at the peak, so there is only $\boxed{1}$ point.

10) Let I_1 be the range of real values that k_1 can take on so that the function $f(x) = x^2 - k_1x + 1$ has no real roots and let I_2 be the range of real values that k_2 can take on so that the function $g(x) = x^2 - k_2x - 4$ has no real roots. What is $I_1 \cup I_2$?

Solution
$\boxed{A}$. Using the discriminant of $f(x)$ and finding when it is less than 0, we see that it is when

$$k_1^2 < 4 \implies -2 < k < 2$$

On $g(x)$, we quickly discover that $g(x)$ will always have a positive discriminant and therefore two positive roots. Thus, the union of these intervals is all real numbers $\boxed{(-\infty, \infty)}$.

11) Let $r(s)$ be a function that gives the inradius of an equilateral triangle with side length s. Compute the value of $\frac{r(15)}{r(5)}$.

Solution

$\boxed{B}$. The inradius can be shown to vary directly with the side length of the triangle, so the answer is $\frac{15}{5} = \boxed{3}$.

12) A linear function is a function $f(x)$ satisfying

$$f(ax + by) = af(x) + bf(y)$$

for all real numbers a, b, x, y. Which of the following is an example of a linear function?

Solution

$\boxed{E}$. While it may seem that A is linear since it is the graph of a line, a linear function can be shown to always pass through the origin (check $f(0x + 0y)$) so A is not linear, but affine. The rest can be shown easily to be nonlinear.

13) Let $S = \{1, 2, 3, 4\}$. Define f by $f : S \to S$ such that f is one-to-one and maps even integers in S to odd integers in S. How many functions f exist?

Solution

$\boxed{A}$. If f is one-to-one and maps even integers to even integers, then 2 must map to either 2 or 4 and 4 must map to the even integer 2 did not map to. There are 2 ways for this to happen. Additionally, since f is one-to-one, 1 must now map to either 1 or 3 and 3 must map to the integer that 1 did not map to. There are two ways to do this, so the total number of functions f is $2 \cdot 2 = \boxed{4}$.

For questions $14 - 16$, let $f(x) = 4x^3 - 2x^2 - 1$.

14) How many real roots does $f(x)$ have?

Solution

$\boxed{B}$. Using Descartes' rule of signs, we can determine that $f(x)$ has exactly 1 positive root and exactly 0 negative roots. We observe 0 is not a root and see that $f(x)$ has $\boxed{1}$ real root.

To be thorough, we should also check to see if there are any possible triple roots, but observe once again that $f(x)$ is not a perfect cube.

15) Most cubic functions have no inverse function, and $f(x)$ is no exception. Which of the following is a necessary and sufficient condition for a function, say $g(x)$, to have an inverse function $g^{-1}(x)$?

Solution

$\boxed{C}$. $f(x)$ has an inverse if and only if it passes the horizontal and vertical line tests, or is one-to-one. It does not need to be monotonic increasing, it must have exactly 1 distinct real root, and being surjective alone does not guarantee that $f(x)$ is invertible.

16) If $f(x)$ were to have an inverse $f^{-1}(x)$, $f^{-1}(x)$ would be the reflection of $f(x)$ across the line $y = x$. What is the sum of the x coordinates of the intersection between $f(x)$ and its reflection across $y = x$?

Solution

$\boxed{D}$. If $f(x)$ intersects its reflection across $y = x$, it must be when $f(x)$ lies on $y = x$ as well.

$$f(x) = x \implies 4x^3 - 2x^2 - 1 = x \implies (x - 1)(4x^2 + 2x + 1) = 0$$

which is only satisfied at $x = 1$, so the answer is $\boxed{1}$.

17) If the line $f(x) = 3x - 2$ intersects the ellipse $3x^2 + 2y^2 = 12$ at point (a, b) in the first quadrant, then what is the value of $12a - 4b$?

Solution

$\boxed{C}$. Drawing a picture, we can confirm there is an intersection in the first quadrant. Note that if (a, b) is a point of intersection, it must lie on the line $f(x) = 3x - 2$, or $3x - y = 2$. Therefore, $3a - b = 2$ and $12a - 4b = \boxed{8}$.

18) Let a_k be the kth element of a sequence with first term a_1. If

$$S(n) = n^2 2^n$$

gives the sum of the first n elements of this sequence, what is the value of a_4?

Solution

$\boxed{C}$. We can compute the fourth term by taking the sum of the first four terms S_4 and subtracting the sum of the first three terms S_3. This gives $256 - 72 = \boxed{184}$.

19) Let $f(x)$ be a cubic polynomial satisfying

$$f(-x) = -f(x)$$

for all real inputs x. Given that $f(x)$ has a root at $x = 3$, what is the maximum possible value of the sum of the remaining two (possibly non-distinct) roots of $f(x)$?

Solution
$\boxed{C}$. The condition given implies f is an odd function. If $f(x)$ is an odd cubic polynomial, then it is of the form $f(x) = ax^3 + bx$ for real numbers a and b, $a \neq 0$. Therefore, 0 must be a root, and because $f(3) = 0$, $f(-3) = 0$ also by the odd condition. So the two remaining roots of $f(x)$ are actually determined and sum to $0 - 3 = \boxed{-3}$.

20) Let $f(n) = 2n - 3$ and $g(n) = \sum_{i=0}^{n} f(i)$. If $g(n)$ is defined for only non-negative integers n, which of the following is an expression for $g(n)$ written as a polynomial in n?

Solution
$\boxed{C}$. Using the form,

$$g(n) = \sum_{i=0}^{n}(2i - 3) \implies g(n) = 2\sum_{i=0}^{n} i - 3(n + 1)$$

Using the sum of consecutive integers, we get that $g(n) = n^2 + n - 3n - 3 = \boxed{n^2 - 2n - 3}$.

21) Let $f(x) = x^2 - 4x + 6$ and let $g(x)$ be a quadratic function in x that intersects $f(x)$ at $x = 2$. If $g(x)$ has a coefficient of -1 on its quadratic term, what is the minimum possible value of $g(3)$?

Solution
$\boxed{C}$. Given the restriction on the leading term, the quadratic that will minimize $g(3)$ is the reflection of $f(x)$ across $y = 2$, which is $g(x) = 2 - (x - 2)^2$. Thus, the minimum value is $2 - 1 = 1$.

22) If, for a real number a,

$$\frac{a}{a^2 + 1} = \frac{1}{3}$$

then compute the sum of all distinct possible values for

$$\frac{a^3}{a^6 + 1}$$

Solution
$\boxed{A}$. We can do the following:

$$\frac{a}{a^2+1} = \frac{1}{3} \implies \frac{a^2+1}{a} = 3 \implies a + \frac{1}{a} = 3$$

If we do the same for the value we have to compute, we see that we are looking for

$$\frac{1}{a^3+\frac{1}{a^3}} = \frac{1}{(a+\frac{1}{a})(a^2-1+\frac{1}{a^2})} = \frac{1}{(a+\frac{1}{a})((a+\frac{1}{a})^2-3)}$$

which, if we plug in our value for $a+\frac{1}{a}$, is $\boxed{\frac{1}{18}}$.

23) If $f(x) = x^x$, then which of the following is least in value?

Solution
$\boxed{B}$. First note that A and C are in fact equal and less than 1, since they are both equal to $\frac{\sqrt{2}}{2}$. We now need to compare $f(\frac{1}{3})$ and $f(\frac{1}{2})$. If we take both sides to the sixth power, we see that

$$f^6(\frac{1}{3}) = \frac{1}{9} < f^6(\frac{1}{2}) = \frac{1}{8}$$

so $\boxed{f(\frac{1}{3})}$ is minimal.

24) The equation

$$(x-6)(x-2)(x+1)(x+5) = 60$$

has four distinct real solutions. If the greatest of these solutions is m and the least of these solutions is n, then the value $m - n = \sqrt{k}$ for some integer k. What is the value of k?

Solution
$\boxed{A}$. Note that

$$(x-6)(x-2)(x+1)(x+5) = (x^2-x-30)(x^2-x-2) = (x^2-x-16)^2 - 196$$

We can now solve $(x^2-x-16)^2 = 256$. We have two cases to consider, but we realize that since the two cases $x^2-x=0$ and $x^2-x-32=0$ will have the same sum of roots but the later has a root greater than 6, both roots will come from that equation. The difference between the roots of that equation is the value of the square root of the discriminant from the quadratic formula, so the value of k is $1-4(-32) = \boxed{129}$.

25) Let

$$f(x) = \prod_{i=0}^{10} \sum_{j=0}^{i} x^i$$

Compute the coefficient of the x^{54} term of $f(x)$ when expanded completely.

Solution

$\boxed{C}$. Note that the degree of this polynomial is

$$\sum_{i=1}^{10} i = 55$$

so in order to get a term of degree 54, we must choose all the highest power component of each term in the product except for one in which we choose the one with power one less than the highest. There are 10 possible terms that we can get this from (there is no choice in the first term which is just 1), so the coefficient is then 10.

26) Let $f(x) = (x+3)^6$. What is the remainder when $f(49)$ is divided by 100?

Solution

$\boxed{D}$. Note that $f(49) = 52^6 = (50+2)^6$. When we compute the binomial expansion of this quantity, we will see that only the 2^6 contributes to the last two digits. Every other term will either have at least two powers of 50, or in the case of the second to last term, be divisible by 100, and thus will contribute 0 to the remainder when divided by 100. Thus, the answer is $\boxed{64}$.

27) Let $f(8x-3) = 16x^3 - 5x - 3$. If

$$f(x) = ax^3 + bx^2 + cx + d$$

for real values a, b, c, d, then compute the value of $(a+c)(b+d)$.

Solution

$\boxed{B}$. Note here that $f(1) = a+b+c+d$ and $f(-1) = -a+b-c+d$. Now we see that $\dfrac{f(1)+f(-1)}{2} = b+d$ and $\dfrac{f(1)-f(-1)}{2} = a+c$. Thus, what we are really looking for is

$$\frac{f(1)+f(-1)}{2} \cdot \frac{f(1)-f(-1)}{2} = \frac{f^2(1)-f^2(-1)}{4}$$

We get $f(1) = -\frac{7}{2}$ by plugging in $x = \frac{1}{2}$ and $f(-1) = -4$ by plugging in $x = \frac{1}{4}$. Thus, our answer is

$$\frac{\frac{49}{4} - 16}{4} = \boxed{-\frac{15}{16}}$$

28) Let $f(x) = x^3 - 4x - 1$. If the 3 real roots of f are a, b, and c, then compute the value of

$$\frac{a}{a^2-4} + \frac{b}{b^2-4} + \frac{c}{c^2-4}$$

Solution

$\boxed{C}$. Note here that, for r a root of $f(x)$, $0 = r^3 - 4r - 1$. Since $r \neq 0$, we get that $r^2 - 4 = \frac{1}{r}$. This means that, since a, b, c are roots of $f(x)$,

$$\frac{a}{a^2-4} + \frac{b}{b^2-4} + \frac{c}{c^2-4} = a^2 + b^2 + c^2$$

Using that the sum of the roots squared is equal to the value of the square of the sum of the roots minus two times the sum of the roots two at a time, this is equal to $0 - 2 \cdot (-4) = \boxed{8}$.

For questions 29 – 30, let $f(n)$ be a function defined on positive integers n such that $f(1) = 0, f(p) = 1$ for all prime numbers p, and

$$f(mn) = nf(m) + mf(n)$$

for all positive integers m and n.

29) Compute the value of $f(2021)$.

Solution

$\boxed{C}$. Because $2021 = 43 \cdot 47$,

$$f(2021) = 47f(43) + 43f(47) = \boxed{90}$$

since 43 and 47 are prime.

30) Let

$$n = 277945762500 = 2^2 3^3 5^5 7^7$$

Compute the value of $f(n)$ in terms of n.

Solution

$\boxed{A}$. We will attempt to solve for a general form a number $n = {p_1}^{k_1} \cdot {p_2}^{k_2} \cdot \dots \cdot {p_m}^{k_m}$ where all p_i are prime and m is a positive integer. Consider the case when, for $f(xy) = yf(x) + xf(y)$, we let $x = p_1$ and $y = \frac{n}{p_1}$. Using the relation and bases cases, we get that

$$f(n) = \frac{n}{p_1} + p_1 f(\frac{n}{p_1})$$

If $(k_1 \geq 2)$ we do do this once more on $f(\dfrac{n}{p_1})$ we get

$$p_1 f(\frac{n}{p_1}) = p_1 \cdot \frac{n}{{p_1}^2} + {p_1}^2 f(\frac{n}{{p_1}^2}) = \frac{n}{p_1} + {p_1}^2 f(\frac{n}{{p_1}^2})$$

and

$$f(n) = \frac{2n}{p_1} + {p_1}^2 f(\frac{n}{{p_1}^2})$$

The idea is that we can show by induction that if we do this k_1 times, we will get that

$$f(n) = \frac{k_1 \cdot n}{p_1} + {p_1}^{k_1} f(\frac{n}{{p_1}^{k_1}})$$

so that $p_1 \frac{n}{{p_1}^k}$. We then can do this for all primes p_i in this factorization. If we continue, we get that

$$f(n) = \frac{k_1 \cdot n}{p_1} + \frac{k_2 \cdot n}{p_2} + \cdots + \frac{k_m \cdot n}{p_m} + f(1) = n \sum_{i=1}^{m} \frac{k_i}{p_i} + 0 = n \sum_{i=1}^{m} \frac{k_i}{p_i}$$

If we apply this form to the n in our problem we get that this is

$$n(\frac{2}{2} + \frac{3}{3} + \frac{7}{7}) = \boxed{3n}$$