**THETA GEMINI ANSWERS and SOLUTIONS**

## ANSWERS

1) C
2) B
3) B
4) A
5) B
6) D
7) C
8) B
9) A
10) A
11) C
12) D
13) E (1/2)
14) D
15) D
16) A
17) B
18) D
19) C
20) C
21) A
22) B
23) C
24) D
25) D
26) B
27) C
28) A
29) E (240)
30) A

# SOLUTIONS

**1.)C,** to find the altitude we will need to construct a right triangle, but first we need to solve for the side length of our equilateral triangle.

$$Area = \frac{s^2\sqrt{3}}{4}$$

$$16\sqrt{3} = \frac{s^2\sqrt{3}}{4}$$

$$64 = s^2$$

$$s = 8$$

With a side length of 8 on each side of the triangle we drop a perpendicular bisector from one of the vertices. As a result, we are left with two right triangles with a hypotenuse of 8, our altitude, and a side that has been cut in half and therefore has a length of 4. We then realize that the triangle is a 30º-60º-90º and has side lengths $k - k\sqrt{3} - 2k$ and since we know that k = 4 our altitude is $4\sqrt{3}$.

**2.)B,** by the remainder theorem we can set the divisor equal to 0 then evaluate. Then, we plug this number into our dividend. Thus, $x - 2 = 0$, $x = 2$

$$(2)^5 - 4(2)^4 \vdash 6(2)^2 - 8(2) + 12 = 32 - 64 + 24 - 16 + 12 = -12$$

**3.)B,** since we are picking our marbles red, white, and blue in any order we can select this in a total of 3! ways which is $(3)(2)(1) = 6$. Thus, there are a total of 6 combinations to pick a different color each time. Since Alexander is putting the marble back into the bag he is drawing with replacement. Suppose he were to draw in the order red, white, then blue. Thus,

$$\left(\frac{4}{15}\right)\left(\frac{6}{15}\right)\left(\frac{5}{15}\right) = \frac{8}{225}$$

However, we must multiply this probability by 6 as there are 6 different ways to achieve picking a different color each time. Thus, our probability is $\left(\frac{8}{225}\right)(6) = \frac{16}{75}$ and 16 + 75 = 91

**4.)A,** we need to find the probability of getting RRR, WWW, or BBB and when dealing with probability the word or means addition. Thus,

$$\left(\frac{4}{15}\right)\left(\frac{4}{15}\right)\left(\frac{4}{15}\right) + \left(\frac{6}{15}\right)\left(\frac{6}{15}\right)\left(\frac{6}{15}\right) + \left(\frac{5}{15}\right)\left(\frac{5}{15}\right)\left(\frac{5}{15}\right) = \frac{405}{3375} = \frac{3}{25}.$$ 3 + 25 = 28.

**5.)B,** since we are given the three angles in terms of x we need to find a way to solve for x and we do this by knowing the fact that the sum of the angles of any given triangle is always 180°. Thus,

$2x + 21 + 3x + 2 + 5x - 13 = 180$

$10x + 10 = 180$

$100x = 170$

$x = 17$

Now we can plug in 22 for x into each of angles and we find that

$ABC = 2(17) + 21^{o} = 55^{o}$

$CAB = 3(17) + 2^{o} = 53^{o}$

$ACB = 5(17) - 13^{o} = 72^{o}$

And therefore, $53^{o}$ is the smallest angle.

**6.)D,** to solve these kinds of questions set them equal to each other then solve algebraically. Thus,

$\sqrt{18 - 8\sqrt{2}} = a - \sqrt{b}$

$18 - 8\sqrt{2} = a^2 + b - 2a\sqrt{b}$

Split into the following cases

| | |
|---|---|
| $-8\sqrt{2} = -2a\sqrt{b}$ | $a^2 + b = 18$ |
| $b = 2, a = 4$ | $b = 2, a = 4$ |

Since both cases work, we know that a = 4 and b =2 and the sum of which is 6.

**7.)C,** to find the area of an annulus we take the area of the larger circle and subtract it by the smaller one leaving us with the annulus. Thus,

$A = \pi r^2 - \pi r^2$

$A = \pi(6)^2 - \pi(4)^2$

$A = 36\pi - 16\pi$

$A = 20\pi$

**8.)B,** to calculate his total profits we take our earnings and subtract our cost. We find that our profits is

$p(x) = g(x) - f(x) = (5x^2 + 5x + 11) - (6x^2 - 3x - 1)$

$p(x) = -x^2 + 8x + 12$

To find its vertex (the maximum profit) we use $\left(-\frac{b}{2a}, p\left(-\frac{b}{2a}\right)\right)$. $-\frac{b}{2a} = -\frac{8}{2(-1)} = -(-4) = 4.$

$p(4) = -(4)^2 + 8(4) + 12 = -16 + 32 + 12 = 28.$

Thus, Jonathan should sell 4 packs per day making him a profit of 28 per pack. Thus, his total profit is (4)(28) = 112. Therefore, 4 + 112 = 116.

**9.)A,** to solve this question we need to utilize the triangle inequality that states that the longest side of a triangle must be less than the sum of the other two sides. In our case we will let x be the short, medium, and long side of the triangle. Thus,

x + 12 > 21, x > 9

12 + 21 > x, x < 33

x + 21 > 12, x > -9 ( we do not use this inequality as we would result in triangles with negative side lengths and not a triangle at all.)

From the other two we get that 9<x<33. We can then translate this into an arithmetic sequence with first term 10, common difference of 1, and we are trying to evaluate the sum from 10 to 32 which is a total of 32 – 10 + 1 = 23. Thus,

Sum = $n\left(\frac{a_1 + a_n}{2}\right) = 23\left(\frac{10+32}{2}\right) = 483$

**10.)A,**

$$\frac{(6 \text{ students})(28 \text{ hours})}{k \text{ (student hours)}} = 4 \text{ replicas}$$

$$\frac{168 \text{ student hours}}{k \text{ student hours}} = 4 \text{ replicas}$$

$168 = k(4 \text{ replicas})$

$k = 42$

Thus, it takes 42 student hours to create one replica. With this information we can solve the latter half of the problem. Thus,

$$\frac{(18 \text{ students})(x \text{ hours})}{42 \text{ student hours}} = 9 \text{ replicas}$$

$18x = (42)(9)$

$x = 21$

Therefore, it will take 21 hours for 18 students to create the 9 replicas.

**11.)C,** since $z = \ a + bi$ we know that $\bar{z} = a - bi$. Substituting this knowledge, we get

$(z)(\bar{z}) = (a + bi)(a - bi) = \ a^2 - \ b^2$

$z + \ \bar{z} = a + bi + a - bi = 2a$

$2a = \ -18$

$a = -9$

$(z)(\bar{z}) = \ a^2 + \ b^2$

$82 = (-9)^2 + \ b^2$

$82 = 81 + \ b^2$

$1 = \ b^2$

1 = b

$a + b = \ -9 + 1 = \ -8$

**12.)D,** we must realize that the sequence is a geometric one with a common difference of $\frac{1}{e}$ as for every term is being multiplied by $\frac{1}{e}$. To find the sum of a convergent geometric sequence we use

Sum = $\frac{a_1}{1-r} = \ \frac{e^2\pi}{1-\frac{1}{e}} = \ \frac{e^2\pi}{\frac{e-1}{e}} = \ \frac{e^3\pi}{e-1}$

**13.)E,** firstly factor both the numerator and denominator. We get

$$f(x) = \frac{(x)(x-2)(2x+1)}{(x)(2x+1)} = x - 2$$

However, we see that when we plug in 0 and $\frac{-1}{2}$ we get $\frac{0}{0}$ which is undefined. Thus, the holes of discontinuity occur at $x = 0$ and $x = \ \frac{-1}{2}$. The sum of which is $\left|\frac{-1}{2}\right| = \frac{1}{2}$.

**14.)D,** we must first find the radius of the sphere. Thus,

$SA = 4\pi r^2$

$144\pi = 4\pi r^2$

$36 = \ r^2$

$r = 6$

Since the cylinder is circumscribed the height of the cylinder is $2r$ and its base has a radius of $r$. Thus, to find its volume we substitute $2r$ for $h$. Thus,

$V = \ \pi r^2 h$

$V = \pi(6)^2(2)(6)$

$V = 432\pi$

**15.)D,** Let A = Chad's age and I = Chris' age. We need to create two different equations from the information given. Thus,

$A + 4 = 2(I + 4)$

$A = 2(I + 4) - 4$

$A - 6 = 3(I - 6) \rightarrow A = 3(I - 6) + 6$

$2(I + 4) - 4 = 3(I - 6) + 6$

$2I + 8 - 4 = 3I - 12$

$I = 16$

$A + 4 = 2((16) + 4)$

$A + 4 = 2(20)$

$A + 4 = 40$

$A = 36$

The units digit of 36 is 6.

**16.)A,** to find the product of the solutions using Vieta's formula we use $\frac{\text{constant}}{a}$ when the highest degree is even and $-\frac{\text{constant}}{a}$ when it is odd. Since our highest degree is even we find that $\frac{64}{4} = 16$. Thus, 16 is the product of the roots.

**17.)B,**

$\log_2(x+1) - \log_4(x) = 2$

$log_4(x+1)^2 - log_4(x) = 2$ (by change-of-base formula)

$log_4\left(\frac{(x+1)^2}{x}\right) = 2$

$\frac{(x+1)^2}{x} = 16$

$x^2 + 2x + 1 = 16x$

$x^2 - 14x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-14) \pm \sqrt{(-14)^2 - (4)(1)(1)}}{2(1)}$

$x = \frac{14 \pm \sqrt{196 - 4}}{2}$

$x = \frac{14 \pm \sqrt{192}}{2}$

$x = \frac{14 \pm 8\sqrt{3}}{2}$

$x = 7 \pm 4\sqrt{3}$

7 + 4 + 3 = 14

**18.)D,** Since the median is 9 we need to test for when $x$ is less than 9 equal to 9 or greater than 9. If we test for when x is less than 9 we find that the data set from least to greatest is 5,7,x,9,9,11 and if x is less than 9 our median is not 9 and therefore x is not less than 9. If x is 9, we get the values 5,7,9,9,9,11 the median of which is obviously 9. So x = 9 is a solution. If we plug in 10 we find 5,7,9,9,10,11 the median is also 9 so x = 10 works. If we plug in 11 the same is also true as it does not affect either the median nor the range. Thus, the possible values for x is 9,10,11 and the sum of which is 30.

**19.)C,** to find the slope between two points use $\frac{y_2-y_1}{x_2-x_1}$. Thus,

$$\frac{y_2 - y_1}{x_2 - x_1} = k$$

$$\frac{6-k}{4-2} = k$$

$$\frac{6-k}{2} = k$$

$$2k = 6 - k$$

$$3k = 6$$

$$k = 2$$

**20.)C,**

I is true as the area of a circle is $\pi r^2$ and if both $r$'s are equal then the area must be equal.

II is true as no matter what all eccentricities of circles are the same. They are always 0.

III is false as two different circles can have the same center but have different radii.

**21.)A,** since the line is perpendicular then the slope of the line is its negative reciprocal. Thus, our slope is $\frac{-1}{3}$. Now we can plug in our coordinate pair as the line must contain this point. Thus,

$$y = mx + b$$

$$2 = \frac{-1}{3}(9) + b$$

$$2 = -3 + b$$

$$b = 5$$

$$y = \frac{-1}{3}x + 5$$

**22.)B,** since the area of the innermost circle is $16\pi$ then its radius is 4 which is one half of a side of the square and therefore the square has a side length of 8. The diagonal of that square is the same as the diameter of the outmost circle. The diagonal of a square is $s\sqrt{2}$ where $s$ is the side length. Thus, the diameter of the outmost circle is $8\sqrt{2}$ and therefore its radius is half of that which is $4\sqrt{2}$. The area of this outmost circle is $\pi(4\sqrt{2})^2 = 32\pi$

**23.)C,** to solve these kinds of questions we take the total amount of letters and call it $n$. We then take the total amount of different repeating letters, and in this case it is the two i's. Thus,

$$\frac{n!}{2!} = \frac{6!}{2!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = 6\cdot5\cdot4\cdot3\cdot1 = 360$$

**24.)D,** according to the fundamental theorem of algebra a polynomial will have $n$ roots where $n$ is the highest degree of the polynomial. In our case, $n = 7$ because of the term $7x^7$.

**25.)D** the area of the segment = area of sector - area of the triangle

segment = $\left(\frac{120\pi}{360}\right)(6)^2 - \left(\frac{1}{2}\right)(6\sqrt{3})(3) = 12\pi - 9\sqrt{3}$

**26.)B,** let $x = 2021$. Now, we can rewrite the determinant as $\begin{vmatrix} x & x-1 \\ x+1 & x \end{vmatrix}$. Solving this we get:

$(x)(x) - (x-1)(x+1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$

**27.)C,** firstly we need to calculate the total amount of fist bumps and high fives which is

$\binom{33+7}{2} = \binom{40}{2} = \frac{40!}{2!\cdot 38!} = 20 \cdot 39 = 780$. Now, we can calculate the total amount of fist bumps which is when only the 33 people are greeting each other which is $\binom{33}{2} = \frac{33!}{2!\cdot 31!} = 33 \cdot 16 = 528$. By the compliment principle we know that there are 780 – 528 high fives as there can be only fist bumps or high fives. Thus, there are 252 high fives. The positive difference of 528 and 252 is 276.

**28.)A,**

I is false as they are congruent and not supplementary

II is true by definition

III is false as they are supplementary and not congruent

**29.)E,** the constant term of this expansion occurs when we have $(x^2)^2 \cdot \left(\frac{-2}{x}\right)^4$ as the x's will cancel out leaving us with $\binom{6}{4} \cdot (-2)^4 = \frac{6!}{2!\cdot 4!} \cdot 16 = 15 \cdot 16 = 240$

**30.)A,** 50% of the time Daniel will choose a integer from the interval [2022, 4042] and in these cases no matter what Matthew chooses he will lose. The other 50% of the time Daniel will choose a integer from the interval [0,2021] and since both of their intervals are mirrored Daniel will win 50% of the time. Thus, the probability of Daniel winning is (0.5)(1) + (0.5)(0.5) = .75 and by the complement principle Matthew's probability of winning is .25 and as a fraction is $\frac{1}{4}$. 1 + 4 = 5