Nationals 2015 Alpha Analytic Geometry – ANSWERS

(1)	C
(2)	A
(3)	D
(4)	A
(5)	B
(6)	E
(7)	D
(8)	C
(9)	A
(10)	D
(11)	C
(12)	B
(13)	A
(14)	C
(15)	D
(16)	E
(17)	A
(18)	B
(19)	D
(20)	C
(21)	D
(22)	B
(23)	A
(24)	B
(25)	A
(26)	C
(27)	B
(28)	D
(29)	C
(30)	C

Nationals 2015 Alpha Analytic Geometry - SOLUTIONS

(1) **Solution:** $\left(3\hat{\imath} + 2\hat{\jmath} - \hat{k}\right) \cdot \left(\hat{\imath} - \hat{\jmath} - \hat{k}\right) = (3)(1) + (2)(-1) + (-1)(-1) = 2 \neq 0$. All of the rest have dot product zero. C

(2) **Solution:** $9x^2 + 4y^2 - 90x + 16y + 205 = 0 \to 9(x^2 - 10x) + 4(y^2 + 4y) + 205 = 0 \to 9(x^2 - 10x + 25) + 4(y^2 + 4y + 4) + 205 = 225 + 16 \to 9(x-5)^2 + 4(y+2)^2 = 36 \to (5, -2)$. A

(3) **Solution:** The standard form of a polar conic is $r(\theta) = \frac{ed}{1+e\cos(\theta)}$ where e is the eccentricity and d is the distance from the focus at the origin to the directrix. So $r(\theta) = \frac{6}{3+2\cos(\theta)} = \frac{2}{1+\frac{2}{3}\cos(\theta)} \to e = \frac{2}{3} \to Ellipse$. D

(4) **Solution:** Via the Shoelace Formula, $A = \frac{1}{2}|(1)(2) + (2)(3) + (0)(-1) + (-1)(0) - (1)(-1) - (-1)(3) - (0)(2) - (2)(0)| = \frac{1}{2}|2 + 6 + 1 + 3| = 6$. A

(5) **Solution:** Two vectors lying in the plane are $\vec{v} = \langle 1,1,1 \rangle - \langle 0,2,1 \rangle = \langle 1,-1,0 \rangle$ and $\vec{w} = \langle 1,1,1 \rangle - \langle 1,0,2 \rangle = \langle 0,1,-1 \rangle$. The normal to the plane can be gotten via the cross product: $\vec{v} \times \vec{w} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k} = \langle 1,1,1 \rangle$. So the equation of the plane is $1(x-1) + 1(y-1) + 1(z-1) = 0 \to x + y + z - 3 = 0$. Plugging in $(3,1,Q)$ gives $3 + 1 + Q - 3 = 0 \to Q = -1$. B

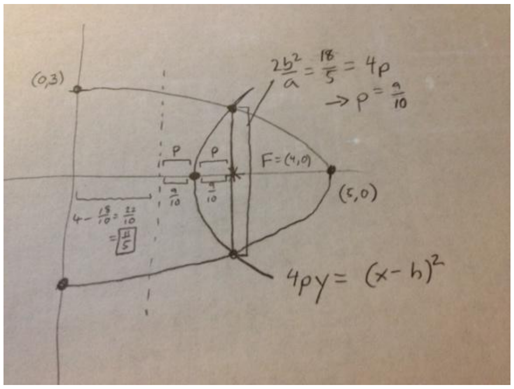
(6) **Solution:** Via the Cosine Double Angle Formula, $r(\theta) = 1 - 2\sin^2(3\theta) = \cos(6\theta)$. For n even, $r(\theta) = \cos(n\theta)$ is a Rose with $2n = 12$ petals. E

(7) **Solution:** $[x \quad y \quad 1]\begin{bmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [x \quad y \quad 1]\begin{bmatrix} Ax + \frac{B}{2}y + \frac{D}{2} \\ \frac{B}{2}x + Cy + \frac{E}{2} \\ \frac{D}{2}x + \frac{E}{2}y + F \end{bmatrix} = Ax^2 + \frac{B}{2}xy + \frac{D}{2}x + \frac{B}{2}xy + Cy^2 + \frac{E}{2}y + \frac{D}{2}x + \frac{E}{2}y + F = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. The others don't work. D

(8) **Solution:** $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$ and $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin(\theta)$ so $\frac{\|\vec{v} \times \vec{w}\|}{\vec{v} \cdot \vec{w}} = \tan(\theta) = \frac{2}{5}$. Therefore $\sec^2(\theta) = 1 + \tan^2(\theta) = \frac{29}{25} \to \cos(\theta) = \frac{5}{\sqrt{29}}$ and $\sin(\theta) = \tan(\theta)\cos(\theta) = \frac{2}{5}\frac{5}{\sqrt{29}} = \frac{2}{\sqrt{29}}$. Finally $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{5}{\sqrt{29}}\frac{2}{\sqrt{29}} = \frac{20}{29}$. C

(9) **Solution:** The center of the ellipse is at the origin, and the foci are at $\left(\pm\sqrt{25-9}, 0\right) = (\pm 4, 0)$. The length of a latus rectum of the ellipse is $\frac{2b^2}{a} = \frac{18}{5}$. If p is the distance between the vertex and the focus/directrix of the parabola, then the length of the latus rectum of the parabola is $4p$. Since the latera recta correspond, this means $4p = \frac{18}{5} \to p = \frac{9}{10}$. Also since the latera recta correspond, the focus of the parabola corresponds to one of the foci of the ellipse. Finally the directrix of the parabola is a

distance of $2p$ from the focus, so the directrix is $4 - 2\frac{9}{10} = \frac{11}{5}$ from the center. See attached picture; any other orientation of the parabola will result in a greater or equal distance from the directrix to the center. A



(10) **Solution:** The determinant of $\begin{bmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{bmatrix}$ is zero for a degenerate conic. For $x^2 + 3xy + 2y^2 + y - 1 = 0$ this determinant is $\begin{vmatrix} 1 & \frac{3}{2} & 0 \\ \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{vmatrix} = -2 - \frac{1}{4} + \frac{9}{4} = 0$. The others are nonzero. Alternatively, you could notice that $x^2 + 3xy + 2y^2 + y - 1 = 0 \to (x + 2y + 1)(x + y - 1) = 0$ is two crossed lines. D

(11) **Solution:** The polar point $\left(\pi, \frac{2\pi}{3}\right)$ is the Cartesian point $\left(\pi\cos\left(\frac{2\pi}{3}\right), \pi\sin\left(\frac{2\pi}{3}\right)\right) = \left(-\frac{\pi}{2}, \frac{\pi\sqrt{3}}{2}\right)$. The distance is therefore

$$D = \sqrt{\left(\pi + \frac{\pi}{2}\right)^2 + \left(\frac{2\pi}{3} - \frac{\pi\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9\pi^2}{4} + \frac{4\pi^2}{9} + \frac{3\pi^2}{4} - \frac{\pi^2 2\sqrt{3}}{3}} = \pi\sqrt{\frac{12}{4} + \frac{4}{9} - \frac{2\sqrt{3}}{3}} = \pi\sqrt{\frac{108}{36} + \frac{16}{36} - \frac{24\sqrt{3}}{36}} = \frac{\pi}{3}\sqrt{31 - 6\sqrt{3}}.$$ C

(12) **Solution:** The descriminant $B^2 - 4AC = 1^2 - 4(1)(2) = -7 < 0$. Therefore this conic is an ellipse; since there is a nonzero xy term it is not a circle. The eccentricity of an ellipse is between zero and one. B

(13) **Solution:** $\cot(2\theta) = \frac{A-C}{B} = \frac{1-2}{1} = -1 \to 2\theta = -\frac{\pi}{4} \to \theta = -\frac{\pi}{8}$. A

(14) **Solution:** The three invariants under conic rotation are $B^2 - 4AC = {B'}^2 - 4A'C'$, $A + C = A' + C'$, and $F = F'$. In this case, $B^2 - 4AC = 7 = (0)^2 - 4A'C' \rightarrow A'C' = -\frac{7}{4}$, $A + C = 3 = A' + C'$, and $F = 8 = F'$. Therefore $\frac{A'+A'C'+C'}{F'} = \frac{3-\frac{7}{4}}{8} = \frac{5}{32}$. C

(15) **Solution:** $Area = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{24(24-15)(24-16)(24-17)} = \sqrt{24(9)(8)(7)} = 24\sqrt{21}$. D

(16) **Solution:** The directional vector of the line is $\vec{v} = \langle 1,2,3\rangle - \langle -1,0,2\rangle = \langle 2,2,1\rangle$. Therefore the parametric equations for the line are $x(t) = 2t + 1, y(t) = 2t + 2, z(t) = t + 3$. When $z = 0$, $t = -3 \rightarrow x = -5 = p$ and $y = -4 = q$. So the answer is -9. E

(17) **Solution:** All that is necessary is to find the sine of the angle between the normal vector to the horizontal plane $\hat{z} = \langle 0,0,1\rangle$ and the normal vector to the cutting plane $\vec{n} = \langle \sqrt{7}, 1, \sqrt{17}\rangle$. We know that $\hat{z} \cdot \vec{n} = \sqrt{17} = \|\hat{z}\|\|\vec{n}\| \cos(\alpha) \rightarrow \cos(\alpha) = \frac{\sqrt{17}}{\sqrt{7+1+17}} = \frac{\sqrt{17}}{5} \rightarrow \sin(\alpha) = \sqrt{1 - \frac{17}{25}} = \sqrt{\frac{8}{25}} = \frac{2\sqrt{2}}{5} \rightarrow e = \frac{\sin(\alpha)}{\sin(\beta)} = \frac{\frac{2\sqrt{2}}{5}}{\frac{\sqrt{2}}{2}} = \frac{4}{5}$. A

(18) **Solution:** The asymptotes will have slope of $\pm\frac{2}{3}$ and go through the center $(0,1)$. This leads to the linear equation $y = \pm\frac{2}{3}x + 1 \rightarrow 2x \mp 3y \pm 3 = 0$. Since all of the answers are in polar form, using $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives $2r\cos(\theta) \mp 3r\sin(\theta) \pm 3 = 0 \rightarrow r = \frac{\pm 3}{2\cos(\theta) \pm 3\sin(\theta)}$. Only B corresponds to this.

(19) **Solution:** The eccentricity is the ratio of the distance from the vertex to the focus to the distance from the vertex to the directrix. Therefore the vertex will be $\frac{2}{5}$ the distance from the focus to the directrix. The equation of the line coincident with the major axis is $y = \frac{1}{2}x$ since it must go through the focus and be perpendicular to the directrix. This line intersects the directrix when $-2x + 3 = \frac{1}{2}x \rightarrow x = \frac{6}{5} \rightarrow y = \frac{3}{5}$. The point $\frac{2}{5}$ the distance from the origin to $\left(\frac{6}{5}, \frac{3}{5}\right)$ is $\left(\frac{2}{5} \cdot \frac{6}{5}, \frac{2}{5} \cdot \frac{3}{5}\right) = \left(\frac{12}{25}, \frac{6}{25}\right)$. D

(20) **Solution:** The volume is the triple scalar product $|\langle 1,1,2\rangle \cdot (\langle 0,2,1\rangle \times \langle 3,1,0\rangle)| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} = |0 + 3 + 0 - 12 - 1| = 10$. C

(21) **Solution:** Eliminating the parameter gives $1 = \sin^2(2t) + \cos^2(2t) = \left(\frac{x+2}{5}\right)^2 + \left(\frac{y-3}{7}\right)^2$. This is an ellipse with semi-axes 5 and 7, so the area is 35π. D

(22) **Solution:** $\frac{3x^2+5x-2}{x+1} = \frac{(3x+2)(x+1)-4}{x+1} = 3x + 2 - \frac{4}{x+1}$. So $y = 3x + 2$. B

(23) **Solution:** $y = \sin(5x) + \sin(3x) = 2\sin(4x)\sin(x)$ via sum-to-product. The period of the product of periodic functions is the maximal individual period, which in this case is 2π. A

(24) **Solution:** $Amplitude = \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{13}\right)^2} = 4$. B

(25) **Solution:** $= \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}} = \frac{|(4)(3)+(-1)(-1)+(3)(2)+2|}{\sqrt{4^2+(-1)^2+3^2}} = \frac{21}{\sqrt{26}} = \frac{21\sqrt{26}}{26}$. A

(26) **Solution:** The graph will have a vertical asymptote when $e^x - e^{-x} = 0 \rightarrow x = 0$. $\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = 1$ and $\lim_{x\to-\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = -1$ so it has two horizontal asymptotes. C

(27) **Solution:** This is the equation of a circle with **diameter** 2. So the area is π. B

(28) **Solution:** We will call the point on the rectangle that lies on the hypotenuse of the triangle (x, y). Clearly the hypotenuse corresponds to the line $y = -\frac{k}{h}x + k$ and the area of the inscribed rectangle is $A = xy$. Therefore $A = x\left(-\frac{k}{h}x + k\right) = -\frac{k}{h}x^2 + kx$. This is a downward opening parabola, which is maximal at its vertex, which always occurs at $x = -\frac{b}{2a} = -\frac{k}{-2\frac{k}{h}} = \frac{h}{2} \rightarrow y == -\frac{k}{h}\frac{h}{2} + k = \frac{k}{2} \rightarrow A = \frac{hk}{4}$. D

(29) **Solution:** This problem describes an ellipse with focus at $(0,2)$, directrix at the x-axis, and eccentricity $e = \frac{1}{2}$, and is asking for the second focus. Since the eccentricity is the ratio of the distance from the vertex to the focus to the distance from the vertex to the directrix, this puts the vertex at $\left(0, \frac{4}{3}\right)$. This means that $a - c = \frac{2}{3}$. Since $e = \frac{1}{2} = \frac{c}{a}$ also, we get $a - 2c = 0$. Solving the system, we get $c = \frac{2}{3}$. This puts the second focus at $\left(0, 2 + 2 \cdot \frac{2}{3}\right) = \left(0, \frac{10}{3}\right)$. C

(30) **Solution:** These points all lie on a circle of radius $4^{\frac{1}{2015}} \approx 1$, and the polygon almost coincides completely with this circle because there are so many points. So the area will be very close to that of the unit circle, π. C