Answers:

1. B
2. A
3. C
4. C
5. B
6. B
7. A
8. B
9. C
10. C
11. E
12. C
13. A
14. D
15. D
16. A
17. C
18. D
19. C
20. E
21. A
22. A
23. B
24. D
25. C
26. B
27. C
28. B
29. E
30. D

Solutions:

1. If we begin with x llamas, after three days there will be $\frac{4^3}{5^3}x$ lllamas left. Therefore the amount of llamas that left is $x - \frac{4^3}{5^3}x$. The fraction therefore is $1 - \frac{4^3}{5^3} = \frac{61}{125}$. **B**

2. This problem has three factors influencing the system, both faucets and the drain. This can be related with the following equation $\frac{T}{6} + \frac{T}{6} - \frac{T}{4} = 1$. Solving for T will give the time it take when working together. $T = 12$. **A**

3. This is simply the points of the function $f(a) = x^3 + ya^2 + za + w$. We now have the points $(1), f(2), f(3)$, and $f(4)$ therefore $f(5)$ can be found with finite differences. Beginning with 2,6 11, and 20, the differences are 4,5, and 9. The next set of differences is 1, 4. And the constant difference is 3. Therefore backtracking will lead to $f(5) = 36$. **C**

4. Let n be the original population. Therefore $n = x^2, n + 100 = y^2 + 1$, and $n + 200 = z^2$. Hence subtracting the first two equations we obtain $(y - x)(y + x) = 99$. The possibilities are $y - x = 1\ and\ y + x = 99, y - x = 3\ and\ y + x = 33, or\ y - x = 9\ and\ y + x = 11$. From these cases the only possibilities are $(49{,}50), (15{,}18), and\ (1{,}10)$. Testing these with our conditions we see that the only one that fits is $(49, 50)$. Hence $49^2 = 2401$. **C**

5. Connecting the centers of the cannonballs we can obtain a tetrahedron of side length 6. Hence the distance from the top to the plane is the altitude of the tetrahedron two of the radii of the balls. The altitude is found with the formula $A = \frac{s\sqrt{6}}{3} = 2\sqrt{6}$. Adding the two radii, we obtain $2\sqrt{6} + 6$. **B**

6. Begin by giving every employee \$10 to satisfy the condition. Now we must distribute 7 indistinguishable dollars to 7 distinguishable people. Hence we use stars and bars and obtain $\binom{13}{6} = 1716$. **B**

7. This can be minimized with the AM-GM inequality. Splitting up the fraction we can simplify to $9x\sin x + \frac{4}{x \sin x}$. Placing into the inequality we obtain $\frac{1}{2}\left(9x^2\sin^2 x + \frac{4}{x\sin x}\right) \leq 6 \rightarrow$ $9x^2\sin^2 x + \frac{4}{x\sin x} \leq 12$. **A**

8. Given that there were 5 coin flips, there are 32 different possibilities of outcomes. Since he remembers that there was at least one heads flip then this number decreases to 31 (the all tails case is eliminated). Hence, there is a $\frac{1}{31}$ chance that he guessed correctly. **B**

9. Since the width and the length decrease the height must increase by a factor we will call x. Hence the new volume is $V = \frac{x}{4}lwh = lwh$. Therefore, the new height increases by a factor of 4. This means that the percent increase is 300%. **C**

10. If the 7 is the first digit we have 81 options, the second 72 options, and the third again 72 options. Hence, we have 225 different options. **C**

11. The cross product between the two vectors takes the form $=< -12, -30, -26 >$. Hence, if we multiply it by the scalar multiple $.5$ we obtain the vector $< -6, -15, -13 >$.**E**

12. The values of theta such that sine is equal to negative one-half are when theta is equal to $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. Hence, if any theta lies between them it satisfies the inequality. The probability is $\frac{11\pi}{6} - \frac{7\pi}{6} = \frac{\frac{4\pi}{6}}{2\pi} = \frac{1}{3}$. **C**

13. This relation can be satisfied in equation form by writing it as $\frac{x}{30} + \frac{x}{30} + \frac{x}{10} = 3$. Note that we set equal to 3 because they are completing three jobs. Hence, solving we obtain $x = 18$. **A**

14. If we have 1728 cubes then we will have a cube of side length 12. Therefore, using surface area we will have $6(12)^2$ blue sides. Now, we will have $6(10)^3$ orange sides in the middle, $8(3)$ orange sides on the corners, $12(10)(4)$ orange sides on the edges, and $6(5)(10)^2$ on the surfaces. Hence the total ratio is 1 to 11. **D**

15. The line that goes through the points $(0,0)$ and $(3,3)$ has a slope of 1 and is $y = x$. Therefore the perpendicular bisector of this line segment is the dividing line. This line can be found to be $y = -x + 3$. Therefore, this triangle formed by the perpendicular bisector and the square is the probability region not desired. Hence the probability desired is $P = \frac{4-\frac{1}{2}}{4} = \frac{7}{8}$. **D**

16. Since Samantha has a sector, the cone's circumference will be the sectors arc length. Hence, The circumference is equal to $\frac{288}{360} = \frac{x}{2\pi(15)} = 24\pi = 2\pi r \rightarrow r = 12$. In addition, the slant height is the sector's radius. Hence the height of the cone can be found to satisfy a 9, 12, 15 triangle thus $h = 9$. Therefore the volume is $V = \frac{1}{3}\pi(12)^2(9) = 432\pi$. **A**

17. We have 8 different combinations of gender, nationality, and age. We will call American boys a, American men b, American girls c, American women d, foreign boys e, foreign men f, foreign girls g, and foreign women h. Hence we formulate the following equations $a + e = 9, a + c = 5, b + f = 9, a + b + c + d = 14, a + b = 6, g + h = 7$. Adding equations we can find out that $e + f = 12$. Hence the sum of all of them is $a + b + c + d = 14, e + f = 14, and\ g + h = 7$. Therefore, the sum is 33. **C**

18. We begin with 4 ounces of coffee in cup 1 and 4 ounces of cream in cup 2. Step one has us pour 2 ounces of coffee from cup 1 to cup 2, and obtain 2 ounces of coffee in cup 1 and 2 ounces of coffee and 4 ounces of cream in cup 2. In step two, we pour 1 ounce of coffee and 2 ounces of cream from cup 2 to cup 1, and obtain 3 ounces of coffee and 2 ounces of cream in cup 1 with the rest in cup 2. Hence at the end we have 3+2=5 ounces of liquid in cup 1, and out of these 2 ounces is cream. Thus the answer is $\frac{2}{5}$. **D**

19. Applying a basic dot product to the following series we obtain a series in the form $\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)}\right)$. Applying a partial fraction decomposition, we obtain $\frac{1}{k} - \frac{1}{k+3}$. Therefore this telescoping series converges to $\frac{11}{6}$. **C**

20. The period is of length 4. Hence, in 3,600 seconds he takes in 900 breaths. **E**

21. All of the probabilities must add up to be equal to one. Hence, $a + \frac{a}{3} + \cdots = 1 \rightarrow a = \frac{2}{3}$. Therefore the mean is $= 1\left(\frac{2}{3}\right) + 2\left(\frac{2}{9}\right) + 3\left(\frac{2}{27}\right) + \cdots = \frac{3}{2}$. **A**

22. The maximum distance that dog can stretch out on all sides can be found by a triangle with the post and the leash. Using Pythagorean theorem, the dog can stretch out $4\sqrt{2}$ on all sides. Hence the sides have an area of $A = 8\sqrt{2} * 5 = 40\sqrt{2}$ and the ends form a circle of area $A = \pi\left(4\sqrt{2}\right)^2 = 32\pi$. Therefore the total area is $40\sqrt{2} + 32\pi$. **A**

23. Since there are 8 points on the circle we can form polygons by choosing 3 of the 8 points, then 4 of the 8 points, etc. Hence we have $\binom{8}{3} + \binom{8}{4} + \cdots + \binom{8}{8} = 2^8 - \binom{8}{0} - \binom{8}{1} - \binom{8}{2} = 219$ different polygons. **B**

24. If the differences 12 and 4 obey a geometric progression then the common ratio is $1/3$. Hence multiplying by 3 will go backwards. Therefore, two steps backwards gives 108. But this is the difference not the actual temperature so we add 20 degrees to obtain 128 degrees. **D**

25. Apply Pappas' Centroid formula. The area of the region can be found with vectors to be 12, the x coordinate of the centroid can be found to be 4, and the radius of revolution is 3. Hence the volume is $V = 2\pi(3)(12) = 72\pi$. **C**

26. This can be evaluated with two counting steps. The first is if the order is exactly the same and the second is if exactly two people switch. Hence, the probability is $\frac{1+\binom{9}{2}}{9!} = \frac{37}{9!}$. **B**

27. At t=2 seconds we have a triangle that has two sides of 8 and 6 and an included angle of 60 degrees. Hence, law of cosines yields $x^2 = 8^2 + 6^2 - 2(8)(6)\cos 60° = 52$. **C**

28. Forming vectors between the points we obtain the vectors $< -3,4,2 >$ and $< 1,-5,1 >$. Hence the cross product is $< 14,5,11 >$. The plane takes the form $14(x-1) + 5(y+1) + 11(z-2) = 0 \rightarrow 14x + 5y + 11z = 31$. **B**

29. Converting the following into trigonometric form we can see that the radius of the shape formed is going to be $= \left(8\sqrt{2}\right)^{1/6} = 2^{\frac{7}{12}}$. Hence, since this shape is a hexagon, the side length is the same as the radius and the area is $\frac{3\left(2^{\frac{7}{12}}\right)^2\sqrt{3}}{2} = 3(54)^{\frac{1}{6}}$ **E**

30. The spring has an amplitude of $\sqrt{3^2 + 4^2} = 5\ m$. However, this is not the maximum vertical distance travelled because the spring goes both up and down, hence it is twice the magnitude $= 10$. **D**