Alpha Ciphering
2016 Mu Alpha Theta National Convention
Answers

0. -4
1. $\sqrt{1+2c}$
2. $\frac{23}{3}$ (only acceptable answer)
3. $\frac{23}{48}$ (only acceptable answer)
4. $\frac{1}{10}$ or 0.1
5. 254
6. 57
7. -20
8. $\frac{5}{3}$ (only acceptable answer)
9. 2880
10. $\frac{10}{21}$
11. $12\sqrt{3}$
12. 4
13. 4
14. $\frac{1}{74}$

Alpha Ciphering
2016 Mu Alpha Theta National Convention
Solutions

0. $4x^4+16x^3-7x^2-28x=0$ factors into $x\left(x+4\right)\left(4x^2-7\right)=0$, so the only integer roots are 0 and –4, whose sum is –4.

1. The sum of the roots is equivalent to *b*, and $b^2=\sin^2\frac{\pi}{9}+\cos^2\frac{\pi}{9}+2\sin\frac{\pi}{9}\cos\frac{\pi}{9}$.
The product of the roots is *c*, so $c=\sin\frac{\pi}{9}\cos\frac{\pi}{9}$. This gives $b^2=1+2c$, so $b=\sqrt{1+2c}$.

2. $y=\log_{3x-2}\left(\frac{x^2-x-2}{x^2-x-6}\right)=\log_{3x-2}\left(\frac{(x-2)(x+1)}{(x-3)(x+2)}\right)$. The base of the logarithm must be positive but not 1, so $x>\frac{2}{3}$ but $x\neq 1$. The antilog must be positive, so $x\neq -2,-1,2,3$. The domain is therefore $\left(\frac{2}{3},1\right)\cup\left(1,2\right)\cup\left(3,\infty\right)$. The sum of these values is $\frac{23}{3}$.

3. Since we are required to take the transpose of another matrix in order to find the inverse matrix, look for elements $a_{3,1}, a_{3,2}$, and $a_{3,3}$. The cofactors of these elements are +(9 – 8)=1, – (–15 – 4)=19, and +(20+6)=26, respectively. The determinant of the original matrix is 96. The three elements we need are therefore 1/96, 19/96, and 26/96, whose sum is 23/48.

4. Let the roots be *a*/*r*, *a*, and *ar*, where *r* represents the common ratio. The product of these three roots is a^3, and by Vieta's formulas, the product is 27/125. The value of *a* is 3/5. Again by Vieta's formulas, the sum of the roots is 1075/250 or 43/10. $\frac{a}{r}+a+ar=\frac{a\left(1+r+r^2\right)}{r}$. Using our value for *a*, we get that $\frac{1+r+r^2}{r}=\frac{43}{6}$, so $6r^2+6r+6=43r\rightarrow(6r-1)(r-6)=0$. This gives two common ratios, but they product the same three numbers: 1/10, 3/5, and 18/5, the smallest of which is 1/10.

5. This is similar to counting the number of possible responses for a true/false test, but

here we are looking at male/female: $2^1+2^2+...+2^7=254$.

6. The three all-positive integer solutions are (5, 11), (12, 7), and (19, 3). The sum of the *x*- and *y*-values is 57.

7. Knowing that $(1+i)^2=2i$ and $(1-i)^2=-2i$ comes in handy here. The problem then becomes $2i+(2i)(1+i)+(2i)^2+(2i)^2(1+i)-2i+(-2i)(1-i)+(-2i)^2+(-2i)^2(1-i)=-20$.

8. $\sum_{n=1}^{\infty}\frac{4}{n^2+4n+3}\to 4\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+1)}$. Using partial fractions, we have $\frac{1}{(n+3)(n+1)}=\frac{A}{n+3}+\frac{B}{n+1}$, where we find that $A=-\frac{1}{2}$ and $B=\frac{1}{2}$. We can rewrite the problem now as $\frac{4}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$. We quickly see a pattern with the partial sums: $2\left[\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\frac{1}{6}...\right]$ – everything cancels other than $\frac{1}{2}$ and $\frac{1}{3}$. The sum is $2\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{5}{3}$.

9. Divide 6300 by 3, 5, and 7. This gives 2100, 1260, and 900. Then divide divide 6300 by the products of 3, 5, 7 taken two and three at a time. This gives 420, 300, 180, and 60. Subtract the first three quotients from 6300, add the products found two at a time, and then add the 60 which is the product of all three numbers. This takes care of overcounting. The final value is 2880.

10. There are 7! ways for the friends to sit. Then, let the person on the left end sit first. He has five choices of where to sit. The person on the right end will therefore have four choices of where to sit. The other five can sit in any of the remaining seats, which means there are 5! ways for them to arrange themselves. Thus, there are $5\cdot4\cdot5!$ ways for the seven friends to arrange themselves. The probability we are looking for is $\frac{5\cdot4\cdot5!}{7!}=\frac{10}{21}$.

11. The three solutions to the equation are 4 and $-2\pm2i\sqrt{3}$. The magnitude of all these is 4, and they are spaced evenly around the plane, so they are 120 degrees apart. Think of this triangle as being composed of three isosceles triangles with legs of length 4 and vertex angle 120 degrees. The total area is
$$3\left(\frac{1}{2}(4)(4)\sin120°\right)=24\left(\frac{\sqrt{3}}{2}\right)=12\sqrt{3}.$$

12. $\left|x^2+8x+12\right| = f = \left|(x+6)(x+2)\right|$. If graphed, $y=(x+6)(x+2)$ is an upward-facing parabola with vertex (−4, −4). Considering the absolute value, the "vertex" is moved to (−4, 4). Since the rest of the graph outside of the *x*-intercepts does not change, then there are exactly three *x*-values that give a value of 4 in this equation.

13. $$43^2 = 1849 \equiv 3 \bmod 13$$
$$\left(43^2\right)^3 \equiv 3^3 \bmod 13$$
$$43^6 \equiv 3^3 \bmod 13 \equiv 1 \bmod 13$$
$$\left(43^6\right)^2 \equiv 1^2 \bmod 13$$
$$43^{12} \equiv 1 \bmod 13$$
$$43^1 \equiv 4 \bmod 13$$
$$43^{12}43^1 \equiv (1)(4) \bmod 13$$
$$43^{13} \equiv 4 \bmod 13$$

14. $|A| = 74$, and by properties of matrices, $\left|A^T\right| = 74$ and $\left|\left(A^T\right)^{-1}\right| = \dfrac{1}{74}$.