Answers:

1. A
2. A
3. B
4. D
5. A
6. C
7. D
8. C
9. D
10. D
11. A
12. E
13. D
14. A
15. D
16. C
17. A
18. C
19. D
20. D
21. E $(\frac{2}{3})$
22. B
23. D
24. A
25. B
26. C
27. D
28. B
29. E $(-\frac{1}{e^{\pi}} - i)$
30. C

Solutions:

1. $(2i)^{10}-(-2i)^9=2^{10}(-1)-(-2)^9 i=-1024+512i$

2. $a=2\text{cis}(60°)$, $b=\text{cis}(-15°)\Rightarrow\frac{a}{b}=2\text{cis}(75°)$

3. By observation of the first few terms of the sequence, each of the even terms $(n\geq 4)$ becomes $i$ while each of the odd terms becomes $-1+i$. Thus, $|a_{2016}|=|i|=1$.

4. $x^2+1=(2\cos\theta)x\ \Rightarrow\ x^2-(2\cos\theta)x+1=0$. By the quadratic formula, we have $x=\frac{2\cos\theta\pm\sqrt{4(\cos\theta)^2-4}}{2}=\cos\theta\pm\sqrt{-(\sin\theta)^2}=\cos\theta\pm i\sin\theta$. Also, $\frac{1}{x}=\bar{x}=\cos(-\theta)\pm i\sin(-\theta)=\cos\theta\mp i\sin\theta$. So by DeMoivre's formula, $x^m+\frac{1}{x^m}=(\cos(m\theta)\pm i\sin(m\theta))+(\cos(m\theta)\mp i\sin(m\theta))=2\cos(m\theta)$.

5. $A$ and $B$ are endpoints of a diameter of the circle $|z|=2$, so any $C$ with magnitude $=2$ would suffice. In this case, $-1.2-1.6i$ works.

6. $7+(-12)-(7+12i)+(\sqrt{7^2+12^2})^2=181-12i$.

7. The sum of all the roots must be 0 since the coefficient on $x^{2015}$ is 0. The sum of the real roots is also 0 since 1 and $-1$ are the only real solutions to the equation. Thus, the answer is 0.

8. $i(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=i(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2})=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$.

9. By the binomial theorem, $\binom{11}{10}y^1(1+i)^{10}=\frac{11!}{10!1!}(2i)^5 y=(11\cdot 32i)y\ \Rightarrow\ 352i$.

10. Let $z'=x+yi$ so $az'=z$. Then,
$z\cdot\bar{z}=az'a\overline{z'}=a^2(z'\overline{z'})=a^2|z'|^2=a^2(x^2+y^2)=a^2h^2$.

11. $\left|\sqrt{\frac{-1}{10}}-i\right|=\left|\frac{\sqrt{10}}{10}i-i\right|=\left|\frac{\sqrt{10}-10}{10}i\right|=\sqrt{0^2+\left(\frac{\sqrt{10}-10}{10}\right)^2}=\left|\frac{\sqrt{10}-10}{10}\right|=\frac{10-\sqrt{10}}{10}$.

12. Using rules of summations and logarithms, we have:

$$\sum_{n=0}^{101}\ln[(-e)^n]=\sum_{n=0}^{50}\ln[e^{2n}]+\sum_{n=0}^{50}\ln[(-e)^{2n+1}]=\sum_{n=0}^{50}(2n)+\sum_{n=0}^{50}\ln[-e^{2n+1}]$$

$$=50\times 51+\sum_{n=0}^{50}(\ln[e^{2n+1}]+\pi i)=2550+\sum_{n=0}^{50}(2n+1+\pi i)$$

$$=2550+51^2+51\pi i=5151+51\pi i$$

13. Let $x=ae^{i\theta}$ and $y=be^{i\psi}$. Then substitution into our given equations yields:
$a^2e^{i2\theta}=be^{i\psi}\ \Rightarrow\ a^2=b, 2\theta=\psi+2\pi k,\ k\in\mathbb{Z}$. Likewise,
$b^2e^{i2\psi}=ae^{i\theta}\ \Rightarrow\ b^2=a, 2\psi=\theta+2\pi j,\ j\in\mathbb{Z}$. Together with the knowledge that $x\neq y$, these facts tell us that $a=b=1$ and $\theta=\frac{2\pi}{3}, \psi=\frac{4\pi}{3}$ since $2\left(\frac{2\pi}{3}\right)=\frac{4\pi}{3}$ and $2\left(\frac{4\pi}{3}\right)=\frac{8\pi}{3}=\frac{2\pi}{3}+2\pi(1)$. Thus, $x=e^{i\frac{2\pi}{3}}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $y=e^{i\frac{4\pi}{3}}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.
So $x+y=-1$.

14. $\det\begin{bmatrix}316-122i & |316-122i|\\ |316-122i| & 316+122i\end{bmatrix}=\det\begin{bmatrix}z & |z|\\ |z| & \bar{z}\end{bmatrix}=z\bar{z}-|z|^2=0$ since these two quantities are equal. Thus, the determinant of our original matrix is also $0$.

15. Careful application of the rules of complex arithmetic yields:
$(-1-i^{-1})^{-1}\cdot i^{-1}=\left(-1-\frac{1}{i}\right)^{-1}\cdot\frac{1}{i}=(-1+i)^{-1}\cdot(-i)=\frac{-i}{-1+i}=\frac{-i(-1-i)}{(-1+i)(-1-i)}=-\frac{1}{2}+\frac{1}{2}i.$

16. We can simplify our given function to $f(z)=\frac{z^2}{|z|^2}=\frac{z\cdot z}{z\cdot\bar{z}}=\frac{z}{\bar{z}}$ so that we can simply plug in $121-144i$ to yield the answer $\frac{121-144i}{121+144i}$.

17. The first region, $|z|\leq\frac{5\sqrt{\pi}}{\pi}$ is a circle centered at the origin of radius $\frac{5\sqrt{\pi}}{\pi}$. The second region is a square with corners at $1+i$, $1-i$, $-1-i$, and $-1+i$, so it has side length 2 and is contained completely within the circle since the magnitudes of the corners (which are furthest away from the origin) are all $\sqrt{2}<\frac{5\sqrt{\pi}}{\pi}$. If this is not immediately obvious, consider $\sqrt{2}<\frac{5}{\sqrt{\pi}}\ \Leftrightarrow$ $\sqrt{2\pi}<5\ \Leftrightarrow\ \sqrt{6.28\cdots}<\sqrt{6.76}=2.6<5$. So the probability of a dart hitting in the square given that it hits in the circle is simply the ratio of their areas, respectively, i.e.
$P(\text{square}|\text{circle})=\frac{A_{\text{square}}}{A_{\text{circle}}}=\frac{2^2}{\pi\left(\frac{5\sqrt{\pi}}{\pi}\right)^2}=\frac{4}{25}=\frac{16}{100}=16\%.$

18. Let $z = i + \frac{1}{i+\frac{1}{i+\cdots}}$. Then we can observe: $z = i + \frac{1}{z} \Rightarrow z^2 = iz + 1 \Rightarrow z^2 - iz - 1 = 0$. By the quadratic formula, $z = \frac{i \pm \sqrt{(-i)^2 - 4(1)(-1)}}{2(1)} = \frac{i \pm \sqrt{3}}{2} = \frac{1}{2}\left(i \pm \sqrt{3}\right)$. Then, we can use the binomial theorem (or the rows of Pascal's triangle for coefficients) to obtain our answer by cubing $z$ and observing that the value remains the same regardless of the $\pm$, i.e.

$$z^3 = \begin{cases} \frac{1}{8}\left(i+\sqrt{3}\right)^3 = \frac{1}{8}\left(i^3 + 3(\sqrt{3})(i^2) + 3(\sqrt{3})^2(i) + (\sqrt{3})^3\right) = \frac{1}{8}\left(-i - 3\sqrt{3} + 9i + 3\sqrt{3}\right) \\ \frac{1}{8}\left(i-\sqrt{3}\right)^3 = \frac{1}{8}\left(i^3 + 3(-\sqrt{3})(i^2) + 3(-\sqrt{3})^2(i) + (-\sqrt{3})^3\right) = \frac{1}{8}\left(-i + 3\sqrt{3} + 9i - 3\sqrt{3}\right) \end{cases}$$

$$= \begin{cases} \frac{1}{8}(8i) \\ \frac{1}{8}(8i) \end{cases} = i$$

Thus the answer is $i$.

19. Draw a picture to see the solution. "All numbers with magnitude less than or equal to $2\pi$" lie in a circle of radius $2\pi$ about the origin. "...having imaginary part 3" means each number will be of the form $x + 3i$ where $x$ is a real number. Plotting this gives a horizontal line whose tails are cut off by the edge of the circle. Thus, the answer is a horizontal line segment.

20. We can see $0 = Q(\overline{z_1}) = P(\overline{z_1}) - (3-4i) \Leftrightarrow 3 - 4i = P(\overline{z_1}) - Q(\overline{z_1})$. And so $P(\overline{z_1}) = 3 - 4i$. Thus, $P(z_1)P(\overline{z_1}) = (3-4i)^2 = 9 - 12i - 12i - 16 = -7 - 24i$.

21. Putting our numbers in complex polar form, we have $z = \frac{\sqrt{2}}{2}e^{i\left(\frac{-\pi}{4}\right)}$ and $w = \frac{\sqrt{2}}{2}e^{i\left(\frac{5\pi}{4}\right)}$. Then, $z^k + w^k = 0 \Leftrightarrow \left(\frac{\sqrt{2}}{2}\right)^k \left[e^{i\left(\frac{-\pi}{4}\right)(k)} + e^{i\left(\frac{5\pi}{4}\right)(k)}\right] = 0 \Rightarrow e^{i\left(\frac{-\pi}{4}\right)(k)} = -e^{i\left(\frac{5\pi}{4}\right)(k)}$. This will occur when $\cos\left(\frac{-\pi k}{4}\right) + i\sin\left(\frac{-\pi k}{4}\right) = -\cos\left(\frac{5\pi k}{4}\right) - i\sin\left(\frac{5\pi k}{4}\right)$. Using the fact that cosine is an even periodic function and sine is an odd periodic function (with periods $2\pi$), and equating the real parts of each side of the equation (as well as the imaginary parts), we can see that the equation will hold whenever $\frac{-\pi k}{4} = \left(\frac{5\pi k}{4} + \pi\right) + 2\pi n$ for any integer $n$. Solving for $k$ yields $k = -\frac{2}{3} - \frac{4}{3}n$. So $K = \left\{\pm\frac{2}{3}, \pm 2, \pm\frac{10}{3}, \cdots\right\}$, which means $|K| = \left\{\frac{2}{3}, 2, \frac{10}{3}, \cdots\right\}$, and $\min|K| = \frac{2}{3}$.

22. Performing the complex arithmetic in cis-form, we have:

$$\prod_{n=1}^{360} 2^{n(-1)^n}(\cos(n^\circ) + i\sin(n^\circ)) = 2^{-1}\text{cis}(1^\circ)\cdot 2^2\text{cis}(2^\circ)\cdots 2^{-359}\text{cis}(359^\circ)\cdot 2^{360}\text{cis}(360^\circ)$$
$$= 2^{(-1+2)+(-3+4)+\cdots+(-359+360)}\text{cis}(1^\circ + 2^\circ + \cdots 360^\circ) = 2^{180}\text{cis}\left(\frac{360(361)}{2}{}^\circ\right)$$
$$= 2^{180}\text{cis}(180^\circ\cdot 361) = 2^{180}\text{cis}(180^\circ) = -2^{180}$$

23. $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$ which is a real number, so D is false while the others are true statements.

24. Equate the real parts and imaginary parts of each equation to obtain two 2x2 systems of equations, i.e.

$$\begin{cases} 2\text{Re}(x) - \text{Im}(x) + 3\text{Re}(y) + 2\text{Im}(y) = 21 \\ -4\text{Re}(x) + 2\text{Im}(x) + 2\text{Re}(y) - 6\text{Im}(y) = 4 \\ \text{Re}(x) + 2\text{Im}(x) - 2\text{Re}(y) + 3\text{Im}(y) = -6 \\ -2\text{Re}(x) - 4\text{Im}(x) + 6\text{Re}(y) + 2\text{Im}(y) = 32 \end{cases} \Rightarrow$$
$$\text{Re}(x) = 1, \text{Re}(y) = 6, \text{Im}(x) = 1, \text{Im}(\text{y}) = 1$$

So $x = 1 + i$ and $y = 6 + i$, thus $|x + y| = |7 + 2i| = \sqrt{49 + 4} = \sqrt{53}$.

25. Recognizing the cosine/sine values for 15° angles, we have:

$$\left[\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right]^{40} = (\cos(15^\circ) + i\sin(15^\circ))^{40} = (1)^{40}\text{cis}(40\cdot 15^\circ) = \text{cis}(600^\circ)$$
$$= \text{cis}(240^\circ)$$

which lies in quadrant $III$.

26. Every four terms cancel, which leaves the final term, which is $i^{2016} = 1$.

27. Using the cyclic nature of powers of $i$ and grouping together like terms, we have:

$$\sum_{n=1}^{102} ni^n = i + 2i^2 + 3i^3 + 4i^4 + \cdots + 102i^{102}$$
$$= (1 + 5 + 9 + \cdots + 101)i + (2 + 6 + 10 + \cdots + 102)i^2 + (3 + 7 + 11 + \cdots + 99)i^3 + (4 + 8 + 12 + \cdots + 100)i^4$$
$$= \frac{26(1 + 101)}{2}i + \frac{26(2 + 102)}{2}(-1) + \frac{25(3 + 99)}{2}(-i) + \frac{25(4 + 100)}{2}(1)$$
$$= 1326i - 1352 - 1275i + 1300 = -52 + 51i$$

28. Let $z = x + yi$. Then, $(1 + i)z = (1 + i)(x + yi) = (x - y) + (x + y)i$. This quantity is only a real number when the imaginary part is 0, i.e. $x + y = 0 \Leftrightarrow y = -x$ which is a line.

29. $e^{i\frac{\pi}{2}}☺e^{i\pi} = i☺(-1) = i^{-1} - (-1)^i = -i - (e^{i\pi})^i = -i - e^{-\pi} = -\frac{1}{e^{\pi}} - i$. (E. NOTA)

30. We can think of the faces analogously as the $\theta$'s of the roots of unity in their polar form: $1 \cdot e^{i\theta}$, i.e. $\left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$. For the product of the two roots Jim rolled to be wholly in the third quadrant, the sum of their angles must be $\frac{4\pi}{3} + 2\pi k$ for some integer $k$. For each value he rolls on the first roll, exactly one other roll will give him the desired result as we can see:

| **Roll 1** | **Roll 2** | **Sum of $\theta$'s** |
|---|---|---|
| 0 | $\frac{4\pi}{3}$ | $\frac{4\pi}{3}$ |
| $\frac{\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ |
| $\frac{2\pi}{3}$ | $\frac{2\pi}{3}$ | $\frac{4\pi}{3}$ |
| $\pi$ | $\frac{\pi}{3}$ | $\frac{4\pi}{3}$ |
| $\frac{4\pi}{3}$ | 0 | $\frac{4\pi}{3}$ |
| $\frac{5\pi}{3}$ | $\frac{5\pi}{3}$ | $\frac{10\pi}{3} = \frac{4\pi}{3} + \pi$ |

So there are 6 ways he can have a product in the third quadrant out of a total of 36 possible roll combinations. Thus, $P(QIII) = \frac{6}{36} = \frac{1}{6}$.