Alpha Gemini Nationals 2015

Answers:

1. C
2. A
3. A
4. D
5. A
6. C
7. E (-28)
8. B
9. E ($0 \le y < \infty$)
10. D
11. C
12. E (40/41)
13. D
14. A
15. D
16. E (3/2)
17. C
18. B
19. B
20. B
21. D
22. E (undefined)
23. A
24. D
25. C
26. C
27. B
28. A
29. C
30. E (no minimum)

Solutions

1.	C	Using Heron's formula: $6 = \sqrt{\left(\frac{9+x}{2}\right)\left(\frac{9+x}{2}-4\right)\left(\frac{9+x}{2}-5\right)\left(\frac{9+x}{2}-x\right)}$ At this point the easiest solution would be to plug in and check rather than solving.
2.	A	$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
3.	A	The units digits are 4,7,6,5,6,3,6. Once you notice that 4 *5 gives 0 there is no need to continue multiplying.

4.	D	$f(f(x)) = \dfrac{\dfrac{x}{1+2x}}{1+\dfrac{2x}{1+2x}}$ $y = \dfrac{x}{1+4x}$ $x = \dfrac{y}{1+4y}$ $y = \dfrac{x}{1-4x} \; or \; \dfrac{-x}{4x-1}$
5.	A	$\left(\dfrac{7}{3}\right)\left(\dfrac{x^3}{2}\right)\left(\dfrac{4}{\sqrt{x}}\right)^4 =$ $\dfrac{7!}{3!4!}\left(\dfrac{1}{8}\right)(256) =$ 1120
6.	C	$\sin x + 2\sin x\cos x + 3\sin x - 4\sin^3 x =$ $2\sin x\left[\cos x + 2 + 2\cos^2 x - 2\right] =$ $\cos x[2\cos x + 1] = 0$ $\sin x = 0$ $\cos x = 0$ $\cos x = -\dfrac{1}{2}$ $x = p, \dfrac{p}{2}, \dfrac{3p}{2}, \dfrac{2p}{3}, \dfrac{4p}{3}$
7.	E	$22x = 154$ $x = 7$ dot-product is $-4x = -28$
8.	B	Multiply top and bottom by 15x then evaluate at 0. $\lim\limits_{x\to 0}\dfrac{15x-3}{30x-10} = \dfrac{3}{10}$
9.	E	$5 \ge \left(5-2x^2\right) > 0$ $5-2x^2 \ge 1$ $-\sqrt{2} \le x \le \sqrt{2}$ max = approaches ∞ as $5-2x^2$ approaches 0 min = 0

10.	D	$x^2 - 30 - 2kx + 11k = 0$ $\sqrt{11k - 30} = k$ $k^2 - 11k + 10 = 0$ $k = 6, 5$
11.	C	$\frac{n!}{r!} = 1680$ $\frac{n!}{r!(n-r)!} = 70$ $n! = 1680r!$ $1680r! = 70r!(n-r)!$ $\frac{1680}{70} = (n-r)!$ $24 = (n-r)!$ $24 = 4!$
12.	E	Solution must be in Quadrant II and thus positive. $\sec\theta = \frac{hypotenuse}{adjacent}$ $hyp = 41$ $adj = 9$ $opp = \sqrt{41^2 - 9^2} = 40$ $\sin\theta = \frac{opp}{hyp} = \frac{40}{41}$
13.	D	Units digit = ${}_4C_1 = 4$ Tens digit = ${}_8C_1 = 8$ Hundreds digit = ${}_7C_1 = 7$ $7 \times 8 \times 4 = 224$
14.	A	$\sqrt{\left(-6\sqrt{3}\right)^2 + 6^2} = r = 12$ $\tan\theta = \frac{y}{x} = \frac{6}{-6\sqrt{3}} = -\frac{\sqrt{3}}{3}$ $\theta = \frac{5\pi}{6}$
15.	D	Multiply first fraction by 6 on top and bottom to simplify to $\frac{3\sqrt{2} - 2\sqrt{3}i}{3\sqrt{2} + 2\sqrt{3}i}$. Then multiply by conjugate to get $\frac{1}{5} - \frac{2\sqrt{6}i}{5}$. Then $\left(\frac{1}{5}\right)^2 + \left(\frac{2\sqrt{6}}{5}\right)^2 = 1$
16.	E	Set up the proportion $\frac{4-r}{5} = \frac{r}{3}$ and solve to get $r = \frac{3}{2}$.

17.	C	Solve for t and plug in to get $y=\frac{9x^2-24x+16}{2}-3x+4$. $y=\frac{9}{2}x^2-15x+12$ Axis of symmetry is $\frac{-b}{2a}=\frac{-(-15)}{2(4.5)}=\frac{15}{9}=\frac{5}{3}$
18.	B	$perimeter=2pr\times\frac{q}{2p}+2r=rq+2r$ $rq+2r=2pr\times\frac{1}{3}$ $q=\frac{2p}{3}-2$ $Area=\frac{1}{2}r^2q=16\left(\frac{2p}{3}-1\right)$ $\frac{1}{2}r^2\left(\frac{2p}{3}-2\right)=16\left(\frac{2p}{3}-1\right)$ $r=4$
19.	B	$\frac{x^2}{16}+\frac{y^2}{36}=1$ $Area/4=pab/4$ $a=2$ $b=3$ $Area/4=24p/4=6p$
20.	B	Use synthetic division using $x=-\frac{1}{3}$.
21.	D	Trace is the sum of the main diagonals. Inverse = $\begin{bmatrix} \frac{7}{3} & -1 & -1 \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$
22.	E	$\left[\left(\frac{1}{i^3}+i^3\right)^3\right]=\left[\left(\frac{1}{-i}-i\right)^3\right]=\left[(0)^{-3}\right]^3=undefined$
23.	A	$b^2-4ac>0$ $64-4(a+6)(a)>0$ $a^2+6a-16<0$ $(a+8)(a-2)<0$ $a=-7,-6,-5,-4,-3,-2,-1,0,1$ But -6 makes changes the quadratic to a linear function with only 1 root so there are only

		8 roots.
24.	D	$x = 4 + 3t$ $y = -4t$ $x = 4 - \frac{3}{4}y$ $4x + 3y = 16$
25.	C	The normal slope is $\frac{3}{4}$ and will go through the point (4, 0). The equation is $y = \frac{3}{4}(x - 4)$ $y = \frac{3}{4}(-4) = -3$
26.	C	$\frac{\log(0.04)}{\log(0.25)} = \frac{\log(5^x)}{\log(4)}$ $\frac{\log\left(\frac{1}{5}\right)^2}{\log\left(\frac{1}{2}\right)^2} = \frac{\log(5^x)}{\log(2^2)}$ $\frac{\log(5)^{-2}}{\log(2)^{-2}} = \frac{\log(5^x)}{\log(2^2)}$ $\frac{-2\log(5)}{-2\log(2)} = \frac{x\log(5)}{2\log(2)}$ $x = 2$
27.	B	$a_1 = 20$ $r = 1/5$ $sum = \frac{20}{1 - \frac{1}{5}} = 25$
28.	A	$110_3 = 12$ $121_4 = 25$ $LCM = 300$ $606_7 = 300$
29.	C	$\frac{\csc x(\sec x - \tan x)}{(\sec x - \tan x)^2 + 1} = \frac{\csc x(\sec x - \tan x)}{\sec^2 x + \tan^2 x - 2\sec x\tan x + 1}$ $\frac{\csc x(\sec x - \tan x)}{2\sec^2 x - 2\sec x\tan x} = \frac{\csc x(\sec x - \tan x)}{2\sec x(\sec x - \tan x)} =$ $\frac{1}{2}\cot x$
30.	E	$y = -2x^2 + 5x - 3$ This is a downward facing parabola so there is no minimum.