Alpha Matrices and Vectors Solutions 2015 MAΘ National Convention

1. The determinant is given by (4*(-1)) – (2*3) = **-10 (D)**

2. A singular matrix has a determinant of 0. Calculating the determinant and setting equal to 0, we obtain $b^2 - 12b - 46 = 0$. The sum of all b is B = 12, so 4B = **48 (C)**

3. Simply add the respective entries in each matrix. **(B)**

4. Use Gaussian elimination and substitution to solve this system of linear equations. We find $x = 2, y = 3, z = 2$. $2x - y + 7z = \mathbf{15}$ **(D)**

5. For three points to be collinear, the area of the triangle containing these three points will be zero. Thus, we form the matrix $\begin{bmatrix} 1 & -1 & t \\ 1 & 3 & 8 \\ 1 & t & 2 \end{bmatrix}$ and set its determinant equal to 0. This gives us $t^2 - 11t = 0$, so t = **0, 11 (D)**

6. By definition, an n×n matrix with n distinct nonzero eigenvalues has 2^n square roots. **(B)**

7. $\begin{bmatrix} 3 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 10 & 4 \end{bmatrix} = \begin{bmatrix} 15+30 & 18+12 \\ 25+10 & 30+4 \end{bmatrix}$. **(A)**

8. We construct matrix A = $\begin{bmatrix} -2 & -7 & -12 \\ 1 & -4 & -9 \\ 4 & -1 & -6 \end{bmatrix}$. The sum of these entries is **-36 (A)**

9. The dot product yields $\frac{\sqrt{6}}{4}$+6+147 = $\frac{\sqrt{6}}{4} + \mathbf{153}$ **(B)**

10. The matrix shown is 3 x 3, and thus is square. Its determinant is clearly zero, and thus the matrix is singular. The matrix is also symmetric. Thus, only **I and III** are true. **(B)**

11. The direction vector (-1, 9, 2) denotes the coefficients of t, while the point on the line can be calculated by plugging in 1 for t and computing the necessary values inside the parentheses. **(A)**

12. Form two vectors, and take the cross product to find (A, B, C) in Ax + By + Cz + D = 0. Plug in a point to find the value of D. $\boldsymbol{-6x + 6y - 16z - 2 = 0}$ **(C)**

13. The adjoint matrix is given by the transpose of the matrix of cofactors. Recall, a cofactor, c_{ij}, is computed by multiplying the minor, m_{ij}, by $-1^{(i+j)}$. Calculating the cofactor matrix of $\begin{bmatrix} 1 & 4 & 3 \\ -8 & 9 & 0 \\ 2 & 7 & 6 \end{bmatrix}$ and transposing we obtain $\begin{bmatrix} 54 & -3 & -27 \\ 48 & 0 & -24 \\ -74 & 1 & 41 \end{bmatrix}$. **(B)**

14. We find the characteristic polynomial as $x^3 - 4x^2$, and thus eigenvalues of 0, 0, 4. The corresponding eigenvectors are (-1, 1, 0), (-2, 0, 1) for the repeated eigenvalue of 0 and

(-2, -1, 1) for the eigenvalue of 4. It is easy enough to test the eigenvectors given in the answers by substituting into the equation $A\vec{x} = \lambda\vec{x}$. **(C)**

15. Factor out a 3 and convert to a rotation matrix with θ equal to 60 degrees. When raised to the 5th power, theta becomes 300 degrees. Thus, we find the matrix

$3^5 \begin{bmatrix} cos\,{}^{5\pi}/_3 & -sin\,{}^{5\pi}/_3 \\ sin\,{}^{5\pi}/_3 & cos\,{}^{5\pi}/_3 \end{bmatrix}$. The entry in row 1, column 2 is $\mathbf{\frac{243\sqrt{3}}{2}}$. **(D)**

16. Converting $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ to reduced row-echelon form, we obtain $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the entry in row 2, column 3 is **3 (A)**

17. Recall that $\|u \times v\| = \|u\|\|v\| sin\theta$. Substituting in the vectors and computing the cross product and magnitudes, we obtain $in\theta = \frac{5\sqrt{6}}{\sqrt{319}}$, so we take the reciprocal to obtain $\mathbf{\frac{\sqrt{1914}}{30}}$ **(A)**

18. We square the incidence matrix to find the number of paths of length 2 between the vertices. $G^2 = \begin{bmatrix} 3 & 3 & 2 & 6 \\ 2 & 5 & 2 & 3 \\ 4 & 4 & 5 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. The entry in the first row, fourth column represents the number of walks of length 2 from v_1 to v_4. **6 (D)**

19. We deduce the pattern $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^n = 3^{n-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Thus, substitute 2015 for n to obtain $\mathbf{3^{2014} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$ **(B)**

20. We reduce matrix Z into reduced row-echelon form as follows $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there are two pivot columns, Z has a rank of **2 (C)**

21. The volume of a parallelpiped is given by the triple scalar product $(a \cdot b \times c)$. Form three vectors, take the cross product, dot them, and take the absolute value to obtain **278 (D)**

22. Use the method of computing an inverse by augmenting the matrix with the identity matrix, $(A|I)$. After reducing A into the identity matrix and applying the same row operations to I, we obtain $(I|A^{-1})$. Applying the same concept, if M is reduced to $3I$, then $4I$ would be reduced to (3*4) M^{-1} = $\mathbf{12M^{-1}}$ **(D)**

23. The shortest distance between 2 skew lines is given by $\frac{|(a-c)\cdot(b\times d)|}{\|b\times d\|}$|, where the lines (q_1) and (q_2) are written in parametric forms $a+bs$ and $c+dt$ respectively. Note, (q_1) and (q_2) are $(-2,3,5)+(1,2,3)t$ and $(0,1,4)+(-1,3,7)s$ respectively. After plugging into the distance formula above, we find the distance is $\mathbf{\frac{5\sqrt{6}}{6}}$ **(E)**

24. det(8A) = 8^(# of rows of A) * det(A) = 8^7 * 4 = 2^23, so n = **23 (D)**

25. The area of the triangle is $\frac{1}{2}\begin{vmatrix} 1 & -5 & 1 \\ -2 & 4 & 1 \\ -3 & -7 & 1 \end{vmatrix} = \frac{42}{2} = \mathbf{21}$ **(C)**

26. Compute the inverse matrix using your favorite method, i.e. augmenting with the identity matrix or finding the adjoint matrix and dividing by the determinant. The inverse matrix is $\begin{bmatrix} -{}^{10}/_{9} & {}^{19}/_{9} & {}^{14}/_{9} \\ -{}^{1}/_{3} & {}^{1}/_{3} & {}^{2}/_{3} \\ {}^{2}/_{9} & -{}^{2}/_{9} & -{}^{1}/_{9} \end{bmatrix}$. Thus, the entry in the 3rd row, 2nd column is $-\mathbf{{}^{2}/_{9}}$ **(D)**

27. Since these two planes are parallel, we can find the shortest distance by picking any point on one plane and calculating the shortest distance to the other plane using the "point to plane" distance formula. The point (0, -1, 0) lies on the second plane and plugging into the distance formula we obtain $\left|\frac{4*0+(-2*-1)+4*0-7}{\sqrt{4^2+2^2+4^2}}\right| = \mathbf{\frac{5}{6}}$ **(D)**

28. The area of the parallelogram is the magnitude of the cross product. The cross product is computed in question 29. $\sqrt{13^2+39^2+26^2} = \mathbf{13\sqrt{14}}$ **(B)**

29. We compute the cross product of u and v by computing the following determinant

$$\begin{vmatrix} i & j & k \\ 8 & -2 & 7 \\ 1 & 3 & -4 \end{vmatrix} = \boldsymbol{-13i+39j+26k}\ \textbf{(A)}$$

30. $proj_v u = {}^{(u\cdot v)}/_{\|v\|} * {}^{v}/_{\|v\|} = -1 * \langle 1,3,-4\rangle = \langle 1,3,-4\rangle$ **(B)**