Answers:

1. C
2. D
3. D
4. E $\left(\frac{-\sqrt{2}}{2}\right)$
5. D
6. D
7. A
8. A
9. C
10. B
11. A
12. C
13. C
14. D
15. E $(-i)$
16. A
17. D
18. A
19. C
20. D
21. D
22. C
23. A
24. D
25. B
26. D
27. A
28. B
29. C
30. D

Solutions:

1. $2.01\overline{6} = 2 + \frac{0}{10} + \frac{1}{10^2} + \left(\frac{6}{10^3} + \frac{6}{10^4} + \cdots\right) = 2 + \frac{1}{100} + \frac{\frac{6}{1000}}{1-\frac{1}{10}} = \frac{201}{100} + \frac{6}{1000} \cdot \frac{10}{9} = \frac{201}{100} + \frac{2}{300} = \frac{605}{300} = \frac{121}{60}$. So $AB = (121)(60) = 7260$.

2. When multiplying the fractions from $\frac{1}{6}$ to $\frac{50}{55}$, all numbers cancel out except 1 through 5 in the numerator and 51 through 55 in the denominator. Further cancellations among those numbers yields the answer.

3. $\prod_{n=2}^{1023} \log_n(n+1) = (\log_2 \cancel{3})(\log_{\cancel{3}} \cancel{4}) \cancel{\cdots} (\log_{\cancel{1022}} \cancel{1023})(\log_{\cancel{1023}} 1024) = \log_2 1024 = 10$.

4. $105 + (105 + d) + (105 + 2d) + \cdots + (105 + 5d) = 180(6 - 2)$
$\Leftrightarrow 6(105) + 15d = 720 \;\Leftrightarrow d = 6\;\Rightarrow$ largest angle: $105 + 5(6) = 135°$. $\cos 135° = \frac{-\sqrt{2}}{2}$.

5. By substitution, $(v-2) + (v-1) + v + (v+1) + (v+2) = y^3 \;\Rightarrow 5v = y^3$ and $(v-1) + v + (v+1) = z^2 \;\Rightarrow 3v = z^2$. For this to happen, all factors of $5v$ must be triples and all factors of $3v$ must be doubles. $3^3 \cdot 5^2 = 675$ is the least number which satisfies both conditions.

6. $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} = S$. So by factoring and substitution, we can convert the second sum to: $1 + r^2 + r^4 + r^6 + \cdots = \frac{1}{1-r^2} = \frac{1}{(1-r)(1+r)} = \frac{S}{1+r}$.

7. $\frac{ar^3}{ar} = \frac{6}{2} = 3 = r^2 \;\Rightarrow r = \pm\sqrt{3} \;\Rightarrow a = \pm\frac{2\sqrt{3}}{3}$, so answer choice (A) is a possible first term.

8. $\left|\frac{1}{2} + \frac{1}{2}i\right| = \frac{1}{2} < 1$ so the series converges. Using the formula for an infinite geometric series:
$S = \frac{a}{1-r} = \frac{\frac{17}{2} - \frac{3}{2}i}{1 - \left(\frac{1}{2} + \frac{1}{2}i\right)} = \frac{17-3i}{1-i} = \frac{17-3i}{1-i} \cdot \frac{1+i}{1+i} = \frac{20+14i}{2} = 10 + 7i$.

9. We can first factor out $\frac{1}{7}$: $\sum_{n=1}^{2016}\left(\frac{n}{7} + \frac{n}{14} + \frac{n}{28} + \frac{n}{56} + \cdots\right) = \frac{1}{7}\sum_{n=1}^{2016}\left(n + \frac{n}{2} + \frac{n}{4} + \cdots\right) = \frac{1}{7}\sum_{n=1}^{2016}\left(\frac{n}{1-\frac{1}{2}}\right) = \frac{1}{7}\sum_{n=1}^{2016} 2n = \frac{2}{7}\sum_{n=1}^{2016} n$. We can then substitute the summation for the formula for the sum of the first n natural numbers (with $n = 2016$) to obtain: $\frac{2}{7} \cdot \frac{2016(2017)}{2} = 288 \cdot 2017 = 2^5 \cdot 3^2 \cdot 2017$.

10. Sum of arithmetic sequence: $S = \frac{n(a_1+a_n)}{2}$, so the average is: $\frac{S}{n} = \frac{n(a_1+a_n)}{2n} = \frac{a_1+a_n}{2} = \frac{74+2020}{2} = 1047$.

11. $\log\left(\frac{1}{2} + 1 + \frac{3}{2} + 2 + \cdots + 1008\right) = \log[(1+2+\cdots+2016)/2] = \log\left(\frac{2016(2017)}{2^2}\right) = \log 504 + \log 2017$.

12. $a_{2016} = a_1 + (n-1)d \;\Rightarrow 2016 = -2016 + (2016-1)d \;\Rightarrow d = \frac{4032}{2015}$ which is only slightly greater than 2. Thus, $a_6 > -2016 + (6-1)(2) = -2006$. In reality, $a_6 \approx -2005.995 \ldots$, and so $\lfloor a_6 \rfloor = -2006$. To obtain the number of integral factors of -2006 we can first find the prime factorization of 2006, then add 1 to the exponent of each factor, multiply them together, then finally double the result as such: $2006 = 2^1 \cdot 17^1 \cdot 59^1 \;\Rightarrow 2(1+1)(1+1)(1+1) = 16$ integral factors.

13. Let $g_1 = a, g_2 = ar, g_3 = ar^2, \ldots$ etc. Then, $ar^{n-1} = ar^n + ar^{n+1} \;\Leftrightarrow\; r^{n-1} = r^n + r^{n+1} \;\Leftrightarrow 1 = r + r^2 \;\Rightarrow r = \frac{-1\pm\sqrt{5}}{2}$ by the quadratic formula. Since all of the terms are given to be positive, our answer is $\frac{-1+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$.

14. $M = \left\lfloor\frac{2016}{5}\right\rfloor + \left\lfloor\frac{2016}{25}\right\rfloor + \left\lfloor\frac{2016}{125}\right\rfloor + \left\lfloor\frac{2016}{625}\right\rfloor = 403 + 80 + 16 + 3 = 502$.
$A = \left\lfloor\frac{2016}{7}\right\rfloor + \left\lfloor\frac{2016}{49}\right\rfloor + \left\lfloor\frac{2016}{343}\right\rfloor = 288 + 41 + 5 = 334$.
$\theta = [0-1]^{2016!} = 1$.
And so, $M - A - \theta = 502 - 334 - 1 = 167$.

15. Let $= -2i + \frac{1}{-2i+\frac{1}{-2i+\cdots}}$, then $x = -2i + \frac{1}{x} \Rightarrow x^2 = (-2i)x + 1$. Applying the quadratic formula here, we get: $x = \frac{-2i\pm\sqrt{-4+4}}{2} = -i$.

16. The last digit of any natural number power of 5 is 5 and there are an even number of terms being added, so the last digit of our sum must be 0.

17. By observation, we can see that the x-coordinate of Phil's position forms an infinite geometric series with first term 625 and common ratio $\frac{-1}{25}$. Likewise, the y-coordinate of his position is an infinite geometric series with first term -125 and common ratio $\frac{-1}{25}$. Applying the formula for the sums of these series, we get that his final coordinate will be at $\left(\frac{5^6}{26}, \frac{-5^5}{26}\right)$. Then, the distance to the origin from there will be:

$d = \sqrt{\left(\frac{5^6}{26} - 0\right)^2 + \left(\frac{-5^5}{26} - 0\right)^2} = \sqrt{\frac{5^{12}+5^{10}}{26^2}} = \frac{5^5\sqrt{5^2+1}}{26} = \frac{3125\sqrt{26}}{26}$.

18. $z + 2z + 3z + \cdots + 99z + 100z = z(1 + 2 + 3 + \cdots + 99 + 100) = z \cdot \frac{100(101)}{2} = 5050z = 2 \cdot 5^2 \cdot 101 \cdot z$. And so the smallest z which would make this a perfect square needs to make each power of the prime factorization even. Thus $z = 2 \cdot 101 = 202$.

19. $|r| < 1$ for an infinite geometric series to converge to a finite sum, so we need $|4x - 8| < 1$ which occurs when $4x - 8 < 1$ and $4x - 8 > -1 \Rightarrow \frac{7}{4} < x < \frac{9}{4}$, which is $\left(\frac{7}{4}, \frac{9}{4}\right)$ in interval notation.

20. $\prod_{n=0}^{100}(1 + i)^n = (1 + i)^{0+1+2+\cdots+100} = (1 + i)^{\frac{100(101)}{2}} = (1 + i)^{5050} = (2i)^{2525} = 2^{2525}i$.

21. The sum of the elements in the nth row of Pascal's triangle is 2^{n-1}. We can see that this sum forms a finite geometric series with first term 1, common ratio 2, and 2016 terms. Thus, the sum is: $S = \frac{a(1-r^n)}{1-r} = \frac{1(1-2^{2016})}{1-2} = 2^{2016} - 1$. Now if we're only concerned with the sum of the interior, we need only subtract away the remaining 1's from this sum. There are two 1's on each row for rows 2 through 2016 and one 1 on the first row. Thus the sum of the interior elements is: $(2^{2016} - 1) - 2015(2) - 1 = 2^{2016} - 4032$.

22. We can factor the denominators of this sum to reveal a pattern: $\frac{6}{3} + \frac{6}{8} + \frac{6}{15} + \frac{6}{24} + \cdots = \frac{6}{1\cdot3} + \frac{6}{2\cdot4} + \frac{6}{3\cdot5} + \frac{6}{4\cdot6} + \cdots = \sum_{n=1}^{\infty}\frac{6}{n(n+2)}$. Using partial-fraction decomposition on this, we can separate the sum as follows: $\frac{6}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \Rightarrow 6 = A(n + 2) + Bn \Rightarrow A = 3, B = -3$. Thus, $\sum_{n=1}^{\infty}\frac{6}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{3}{n} - \frac{3}{n+2}\right) = \left(\frac{3}{1} - \cancel{\frac{3}{3}}\right) + \left(\frac{3}{2} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{5}}\right) + \left(\cancel{\frac{3}{4}} - \cancel{\frac{3}{6}}\right) + \cancel{\cdots} = 3 + \frac{3}{2} = \frac{9}{2}$.

23. $a_{10} = a_1 + (10 - 1)d \Rightarrow 47 = a_1 + 9(3) \Rightarrow a_1 = 20$. Since there are 30 days in June, $n = 30$ in the following formula for the sum of the first n terms of an arithmetic sequence: $S_n = \frac{n}{2}[2a_1 + (n - 1)d] = \frac{30}{2}[2(20) + (30 - 1)(3)] = 1905$.

24. $A = 15 + 9(7) = 78$

$B = \frac{-2}{9}$

$C = \frac{10(10+1)(2\cdot10+1)}{6} = 385$

$76 = a_1 + (12 - 1)(7) \Rightarrow D = -1$

Plugging these values into the matrix, we obtain:

$$\begin{vmatrix} 385-78 & \frac{-2}{9} \\ \frac{81}{2} & -1 \end{vmatrix} = -307+9 = -298.$$

25. $6R_1 + 7R_2 + 7R_3 + 7R_4 + 7R_5 = (7 \cdot \text{ sum of the roots}) - R_1 = 7 \cdot \left(\frac{217}{40} \cdot \frac{5}{7}\right) - \frac{1}{8} = 27.$

26. The probability Amy wins can analogously be viewed as the sum of the probabilities that she wins on any one turn. From observation, we can see:

$P(\text{Amy wins on turn } 1) = \frac{1}{6}$ by Amy rolling a 6 directly.

$P(\text{Amy wins on turn } 2) = \left(\frac{2}{3}\right)\frac{1}{6}$ by Amy rolling a 2-5, then Bo rolling a 1.

$P(\text{Amy wins on turn } 3) = \left(\frac{2}{3}\right)^2 \frac{1}{6}$ by Amy rolling a 2-5, Bo rolls a 2-5, then Amy rolls a 6...

This forms an infinite geometric series with first term $\frac{1}{6}$ and common ratio $\frac{2}{3}$. So the probability Amy wins in total is: $\frac{\frac{1}{6}}{1-\frac{2}{3}} = \frac{1}{2}$.

27. Out of the 203 numbers in the sequence, 101 of them are even and 102 of them are odd. Since each number is getting paired up with some number in the sequence and their difference is taken in the product, it is inevitable that at least two odd numbers end up paired together by the pigeonhole principle. Since the difference of any two odd numbers is an even number, at least one even number will appear in the product which guarantees the product will always be even.

28. $x + (x+1) + (x+2) + \cdots + (x+17) = 18x + (1+2+\cdots+17) = 18x + \frac{17(18)}{2} = 18x + 153$. Factoring this sum, we get: $9(2x+17)$. In order for this product to be a perfect square, $2x+17$ must be a perfect square since 9 is already a perfect square. The smallest x which accomplishes this is 4 since $2(4) + 17 = 25$. So the smallest value the sum could be is $9(25) = 225$.

29. If p_n is the nth prime, then $D_n = p_1 p_2 \ldots p_n + 1$ since it can't be divisible by any primes smaller than p_n and must leave a remainder of 1 when divided by any of the first n primes, and this is the smallest number which has those properties. Thus,

$D_1 = 2 + 1 = 3$

$D_2 = (2)(3) + 1 = 7$

$D_4 = (2)(3)(5)(7) + 1 = 211$

$D_5 = (2)(3)(5)(7)(11) + 1 = 2311$. And so our quantity becomes:

$$\frac{D_5 - D_4}{D_2 D_1} = \frac{2311 - 211}{(7)(3)} = \frac{2100}{21} = 100$$

30. The sum of the interior angles of a quadrilateral is $180(4-2) = 360°$. Let $m\angle A = m\angle B = q$. Then $m\angle C = 2q, m\angle D = 3q$. We know that the sum of the first n terms of the Fibonacci sequence is $F_{n+2} - 1$. So using this, we have:

$$m\angle A + m\angle B + \cdots + m\angle D = q(1+1+2+3) = q(7) = 360°$$

And so $q = \frac{360}{7}$, implying that $3q = \frac{1080}{7}$ which is reduced. Thus, $x = 7, \text{and } y = 1080$, so $y - x = 1073$.