

1. **B.** Divide through by 6 to get the eccentricity as .5. Since the numerator of the equation is equal to  $e(p)$  where  $p$  is the focal radius,  $2 = (.5)(p)$  and  $p = 4$ . Since the equation shows a negative cosine, the directrix is located  $p$  units to the left so  $x = -4$ .
2. **C.** Let  $A = \overrightarrow{P_1P_2} = -i + 2k$ ; this is a vector along the line. Then a unit vector along the line is  $u = \left(\frac{1}{\sqrt{5}}\right)(-i + 2k)$ . Let us take  $Q$  to be  $P_1(0, 0, 0)$ . Then  $\overrightarrow{PQ} = -i - 2j + k$ , so we get for the distance  $PR = \frac{1}{\sqrt{5}} |(-i - 2j + k) \cdot (-i + 2k)| = \frac{1}{\sqrt{5}} |-4i + j - 2k| = \sqrt{\frac{21}{5}}$
3. **A.** Examine a fourth of one circle that includes the shaded region. Draw in an isosceles triangle such that the hypotenuse cuts one shaded region in half. Calculate the area of half the shaded region to be  $A = \frac{\pi}{4} - \frac{1}{2}$ . So total area =  $A = 8\left(\frac{\pi}{4} - \frac{1}{2}\right) = 2\pi - 4$
4. **E.** The circle's area is  $50\pi$ . We find the side length of the square to be  $x^2 + 4x^2 = 100$ ,  $x = \sqrt{20}$ ,  $s = 2\sqrt{20}$ . Thus, the area of the square is 100 so the total area is  $50\pi - 80$ .
5. **A.** Set point B as the origin (0,0). Assign an arbitrary side length (ex. 4) to the triangle. Now point C becomes  $\left(2, \frac{2\sqrt{3}}{3}\right)$  and point A becomes  $\left(2, \frac{4\sqrt{3}}{3}\right)$  Create vectors BC and BA to be  $2i + \frac{2\sqrt{3}}{3}j$  and  $2i + \frac{4\sqrt{3}}{3}j$  respectively. Since  $\sin \theta = \frac{|A \times B|}{|A||B|}$ , take the cross product and divide by the magnitude to yield  $\frac{\sqrt{21}}{14}$ .
6. **C.** The tangent of the first angle is  $\frac{5}{12}$  and the tangent of the second angle is  $\frac{4}{3}$ . We can find the slope of the bisector by finding  $\tan \frac{a+b}{2}$ . Using addition formulas followed by a half angle formula, we find this computation to yield  $\frac{7}{9}$  as our positive angle value.
7. **A.** A simple substitution will reveal the only integers in that set to be the point (3,4), which sum to 7.
8. **B.** The perpendicular bisector of a chord always goes through the center. The perpendicular slope is 4 and goes through (4,1) so our equation becomes  $y = 4x - 15$ . The line  $y = 2$  also goes through the circles center due to the tangency. Thus,  $2 = 4x - 15$  and  $= \frac{17}{4}$ .
9. **B.** There are many ways to solve this problem, but the easiest is with trigonometry. We have  $AD \cdot \sin(DAP) = AD \cdot 2 \sin(DAM) \cos(DAM) = 4 \cdot 2 \cdot \frac{2}{\sqrt{20}} \cdot \frac{4}{\sqrt{20}} = \frac{16}{5}$ .

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10. **A.** Draw a diagram to see that the segment of the tangent line between the x and y axes has length  $r\sqrt{2}$  and that the sides of the small triangle satisfies  $r^2 = r + \frac{r^2}{2}$  so  $r=2$ .
11. **D.** Plug in as dictated by the question and compute  $B^2 - 4AC$  to reveal an ellipse.
12. **D.** Rule out answer choice C due to it being a rectangular hyperbola. Test A, B, and D to reveal D as the answer.
13. **B.** The latus rectum is equal to  $\frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$
14. **B.**  $\frac{c}{a} = .5, c = .5(2) = 1$
15. **D.**  $2p = 3, p = \frac{3}{2}$ . Latus rectum is  $4p = 4\left(\frac{3}{2}\right) = 6$ .
16. **C.** The altitude of the regular tetrahedron with side length  $a$  is equal to  $\frac{a\sqrt{6}}{3}$ . Create a triangle with the altitude of the tetrahedron, the height of one equilateral face, and part of the height of the bottom equilateral face. In an equilateral triangle the heights all intersect at the centroid. We want our leg length to be the distance from the relevant side to the centroid of the base triangle, which is one third of the height of that triangle. Thus,  $\frac{a\sqrt{3}}{2}$ . So  $\frac{\sin \theta = \frac{a\sqrt{6}}{3} a\sqrt{3}}{2} = \frac{2\sqrt{2}}{3}$
17. **C.**  $V = \frac{4}{3}\pi abc = \frac{4}{3}\pi(2)(3)(1) = 8\pi$
18. **A.** Convert to rectangular to see that this is the intersection of a double napped cone and a vertical plane, yielding a hyperbola.
19. **E.**  $t = 4 \ln \frac{x}{4}, y = 3e^{4 \ln(\frac{x}{4})}, y = 3\left(\frac{x}{4}\right)^4$
20. **C.** This problem can be done many ways, but most easily it can be simplified to finding the first lattice point after (0,0) that the line  $y = \frac{3}{4}x$  passes through. This point is (4, 3). Thus,  $D = \sqrt{16 + 9} = 5$ .
21. **B.** Use double angle formulas to yield  $\frac{1}{8}\sin(8\theta)$ .
22. **C.** This is an octagon with circumradius 1. Use  $A = \frac{1}{2}(a)(b)\left(\sin C\right) = 1/2(1)(1)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}$ .  
Multiple by the 8 triangles to find the area to be  $2\sqrt{2}$ .

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23. **E.** For OA,  $y_A = x_A$ . For OB,  $y_B = 7x_B$ . We also have  $\sqrt{(x_A)^2 + (y_A)^2} = \sqrt{(x_B)^2 + (y_B)^2}$ .

Substitution reveals  $x_A = 5x_B$ . Thus, for the slope,  $m = \frac{y_B - 5x_B}{x_B - 5x_B} = \frac{7x_B - 5x_B}{-4x_B} = -\frac{1}{2}$

24. **C.** Create a unit cube with one vertex on the origin. Let the diagonal connect the vertices (0, 0, 0) and (1, 1, 1). The equation of the plane then becomes  $x + y + z = \frac{3}{2}$ . This plane intersects edges connecting vertices whose coordinates sum to 3. For example, it intersects the edge connecting (1, 0, 0) to (1, 1, 0) because at a point halfway between these vertices we have  $x + y + z = \frac{3}{2}$ . There are 6 such edges.

25. **C.** Cosine attains a maximum at 1. Adding in the .25 shift upwards gives 1.25.

26. **A.** Draw a picture and apply law of cosines.  $a^2 = 25 + 25 - 2(5)(5)(\cos 120) = 5\sqrt{3}$

27. **E** Apply Pappus' theorem.  $V = 2\pi rA = 2\pi(6)(25\pi) = 300\pi^2$ .

28. **D.** Draw a picture. The lemniscate lies on the x-axis with a horizontal length 5 in both directions. The second equation is a circle with radius 4. They intersect 4 times.

29. **A.** Create two vectors using one point as a starting point. We will start with (2, -1, -1) and create vectors  $-1i + 5j + 1k$  and  $-5i + 5k + 2k$ . Take the cross product to yield  $5i - 3j + 20k$ . Thus,  $5x - 3y + 20z + D = 0$ . Plug in a point to see that  $D = 7$ . Then,  $5x - 3y + 20z + 7 = 0$ .

30. **D.** The side length of the square is equal to  $2c$  so  $c = 2$ . Remember that the sum of the distances from each of the two foci to a point on the ellipse is equal to  $2a$ . Pick a vertex of the square to find that  $2a = 2 + \sqrt{20} = 2 + 2\sqrt{5}$ . Thus,  $a = 1 + \sqrt{5}$ . Since  $a^2 = b^2 + c^2$ ,  $b^2 = 2 + 2\sqrt{5}$ , so  $b = \sqrt{2 + 2\sqrt{5}}$ .  $2b = 2\sqrt{2 + 2\sqrt{5}} = \sqrt{8 + 8\sqrt{5}}$ . Thus, the square of this is  $8 + 8\sqrt{5}$  and  $8 + 8 + 5 = 21$ .