2017 Alpha Ciphering
Mu Alpha Theta National Convention

0. 6
1. -4
2. $\sqrt{10}$
3. -1
4. $-2i$
5. 89
6. 12
7. 3
8. $\frac{\sqrt{3}}{2}$
9. 1431
10. 210
11. $\frac{\sqrt{165}}{2}$
12. $\frac{1}{4}$

0. Simplify $\left(\sqrt[3]{71}-\sqrt[3]{65}\right)\left(\sqrt[3]{5041}+\sqrt[3]{4615}+\sqrt[3]{4225}\right)$.

This is 71–65 factored as a difference of squares. 71–65 = 6.

1. Find the value(s) of a such that the system of equations has no solution.

$$\begin{cases} x+2y-3z=4 \\ 3x-y+5z=2 \\ 4x+y+\left(a^2-14\right)z=a+2 \end{cases}$$

Using row reduction: $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ The system will have no solution if the last row reads 0 0 0 <non-zero number>. This occurs only when $a=-4$.

2. Find the value of a if $\log_a(10)+\ldots+\log_a\left(10^n\right)+\ldots+\log_a\left(10^{10}\right)=110$.

Using properties of logarithms, this simplifies to $\log_a 10^{1+2+3+\ldots+10}=110 \to$
$\log_a 10^{55}=110 \to 55\log_a 10=110 \to \log_a 10=2 \to a^2=10 \to a=\sqrt{10}$.

3. If $\sin x + \cos x = -1$, find the value of $\sin^{2017} x + \cos^{2017} x$.

$(\sin x + \cos x)^2 = (-1)^2 \rightarrow \sin^2 x + \cos^2 x + 2\sin x \cos x = 1 \rightarrow \sin x \cos x = 0$. Now we know that $\sin x = 0$ or $\cos x = 0$. If $\sin x = 0$, then $\cos x = -1$; if $\cos x = 0$, then $\sin x = -1$. Either way, the 2017th powers give us $-1+0=-1$.

4. Simplify $\ln\left[-(i\cos 1 + \sin 1)^2\right]$, where the natural logarithm is defined over the complex numbers and the imaginary part of the value is as close to 0 as possible.

$$\ln\left[-(i\cos 1+\sin 1)^2\right] \rightarrow \ln\left[-\left(-\cos^2 1 + 2i\sin 1\cos 1 + \sin^2 1\right)\right] \rightarrow$$
$$\ln\left[\cos^2 1 - \sin^2 1 - (2\sin 1\cos 1)i\right] \rightarrow \ln[\cos 2 - i\sin 2] \rightarrow \ln[\cos(-2) + i\sin(-2)] \rightarrow$$
$\ln\left[e^{-2i}\right] = -2i$. (other values are coterminal with -2, but this is the closest value to 0)

5. The product of the ages of a group of teenagers is 10584000. What is the sum of their ages?

$10584000 = 2^6 3^3 5^3 7^2$. We can only use ages from 13-19, and the only age here that is divisible by 5 is 15 and the only age divisible by 7 is 14. Dividing by 15^3 and 14^2, we are left with 16. $2(14)+3(15)+16=89$.

6. Find the distance between the vertices of $r = \dfrac{32}{3+5\sin\theta}$.

Since we have sine in the denominator, the hyperbola is vertical. To find the vertices, substitute $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ for q. At $\theta = \dfrac{\pi}{2}, r = 4$; at $\theta = \dfrac{3\pi}{2}, r = -16$. Since we get a negative r-value at $\dfrac{3\pi}{2}$, we move 16 in the direction of $\dfrac{\pi}{2}$. The distance between the points is 12.

7. Find the remainder when 2^{65536} is divided by 13.

$$2^6 \equiv -1 \bmod 13 \rightarrow 2^{12} \equiv \left((-1)^2 = 1\right) \bmod 13 \rightarrow \left(2^{12}\right)^{5461} 2^4 \bmod 13 \rightarrow 1\cdot 16 \bmod 13 \rightarrow$$
$$16 \bmod 13 \rightarrow 3 \bmod 13 \Rightarrow 3.$$

8. Find the value of $\cos\left(\dfrac{\pi}{6}\right) + \ldots + \cos\left(\dfrac{n\pi}{6}\right) + \ldots + \cos\left(\dfrac{2017\pi}{6}\right)$.

Cosine has a period of 2p and the sum of the values in each revolution is 0; that is, $\cos\left(\dfrac{\pi}{6}\right) + \cos\left(\dfrac{2\pi}{6}\right) + \cos\left(\dfrac{3\pi}{6}\right) + \ldots + \cos\left(\dfrac{12\pi}{6}\right) = 0$. After dividing 2017 by 12, we get a remainder of 1, so the expression is equivalent to $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$.

9. The coefficients of the third and eleventh terms of the expansion of $(a+b)^n$ are the same (when the terms are written in descending power of *a* and *n* is a positive integer) . Find the sum of the coefficient of the fifth term and all positive integral divisors of that coefficient.

   Due to symmetry in binomial expansion, if the third and eleventh terms are the same, then there are 13 terms and the exponent on the binomial is 12. The fifth term will be $\frac{12!}{8!4!}a^8b^4$.

   The coefficient simplifies to 495. $495=3^2 5^1 11^1$ and the sum of the positive factors will be $(9+3+1)(5+1)(11+1)=936$. The sum is 1431.

10. If $(a+b+c+d+e+f+g)^5$ is expanded and simplified, how many terms will contain only three letters?

    There are ${}_7C_3=35$ ways to get three letters out of the 7. The exponents of these three letters must be positive and add to 5. Using "stars and bars" there are ${}_{5-1}C_{3-1}=6$ ways to do this. The product of 35 and 6 is 210.

11. Find the area enclosed by the triangle whose vertices are (1, 0, 4), (3, –3, 0), and (0, 1, 2).

    Let *A*(1, 0, 4), *B*(3, –3, 0), and *C*(0, 1, 2). Find the length of each side:
    $AB=\sqrt{2^2+3^2+4^2}=\sqrt{29}, BC=\sqrt{3^2+4^2+2^2}=\sqrt{29}, AC=\sqrt{1^2+1^2+2^2}=\sqrt{6}$.

    Using Heron's formula, we have $A=\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)}=$

    $$\sqrt{\left(\frac{2\sqrt{29}+\sqrt{6}}{2}\right)\left(\frac{\sqrt{6}}{2}\right)\left(\frac{\sqrt{6}}{2}\right)\left(\frac{2\sqrt{29}-\sqrt{6}}{2}\right)}=\sqrt{\left(\frac{4\cdot 29-6}{4}\right)\left(\frac{6}{4}\right)}=\frac{\sqrt{165}}{2}.$$

12. A point is selected at random from inside a circle. Find the probability that the point is closer to the center of the circle than to the circle itself.

    We want the circle with the same center but with half the radius. $P=\frac{\pi\left(\frac{1}{2}r\right)^2}{\pi r^2}=\frac{1}{4}$.