Answers:

1. A
2. B
3. D
4. A
5. B
6. A
7. E
8. B
9. A
10. C
11. C
12. B
13. D
14. B
15. D
16. A
17. E
18. B
19. A
20. C
21. B
22. C
23. E
24. C
25. A
26. E
27. B
28. B
29. C
30. B

Solutions:

1. $f'(1) = f(1-1) = f(0) = 1$. **A**

2. Splitting up the fraction suggests that this differential equation is homogeneous. $\frac{x^2+y^2}{2xy} = \frac{1}{2}(\frac{x}{y} + \frac{y}{x})$. Make the substitution $y = ux$ and simplify. $u'x + u = \frac{1}{2}\left(u + \frac{1}{u}\right) \rightarrow \frac{dx}{x} = \frac{2udu}{1-u^2}$. Integrating both sides results in: $\ln|x| = -\ln|1-u| - \ln|1+u| + C \rightarrow x(1-u^2) = C \rightarrow y^2 = x^2 - Cx$. Plugging in initial condition and simplifying gives $y = x$ (since y is positive). Now, evaluating $y(e)$ gives a final answer of e. **B**

3. Taking a derivative, $g'(x) = f'(x) - 2f'(2x)$. Plugging in the two values we obtain two simultaneous equations. $5 = f'(1) - 2f'(2x)$ and $7 = f'(2) - 2f'(4)$. What we want to obtain is $h'(1) = f'(1) - 4f'(4)$. We can see that this expression can be evaluated by eliminating a variable in our system of equations. Multiplying the second equation by 2 and adding yields: $f'(1) - 4f'(4) = 19$. **D**

4. The key to this problem is realizing that the left side of the equation is simply the derivative of the product. It easily simplifies to: $(\sin t * y)' = \sec^2 t$. Integrating both sides goes to: $y \sin t = \tan t + C$. Plugging in the initial value the equation now becomes $y(t) = \sec t$. Plugging in the desired value the answer is -1. **A**

5. The integral that is precisely placed in the problem suggests that the second fundamental theorem of calculus may be useful. Deriving both sides gives: $2f(x)f'(x) = f(x)^2 + f'(x)^2$. This is simply the expansion of $\left(f(x) - f'(x)\right)^2$. The equation now becomes $f(x) = f'(x)$. Separating and solving gives $f(x) = Ce^x$ as the set of all solutions. Using the initial value it becomes $f(x) = e^x$. Finally, $f(1) = e$ which is closest to 3. **B**

6. Cross multiplying this differential equation insists that $N(x,y) = 6ye^x - x^2$ and $M(x,y) = -2xy + \sin x + 3y^2e^x$. Hence, $\frac{\partial M}{\partial y} = 6ye^x - 2x = \frac{\partial N}{\partial x}$ and the equation is exact. **A**

7. Since this equation is exact, we can begin by integrating $F(x,y) = \int M(x,y)dx + h(y)$. $F(x,y) = \int(-2xy + \sin x - 3y^2e^x)dx + h(y) = -x^2y - \cos x + 3y^2e^x + h(y)$. Taking the derivative of our newly found F should be equal to $N(x,y)$. $\frac{\partial F}{\partial y} = N(x,y) = 6ye^x - x^2 = -x^2 + 6ye^x + h'(y) \rightarrow h(y) = C$. Hence, our solution is of the form $= -x^2y - \cos x + 3y^2e^x = C$. Plugging in the initial value gives a value of $C = 11$. **E**

8. $M(x,y) = 2x\cos y - x^2$ and $N(x,y) = x^2\sin y$. $\frac{\partial M}{\partial y} = -2x\sin y, \frac{\partial N}{\partial x} = 2x\sin y$. Not exact.
B

9. This is a differential equation that is proportional to the amount present. Hence: $\frac{dy}{dt} = ky$. $\int \left(\frac{1}{y}\right) dy = kdt \to y = Ce^{kt}$. Plugging in $t = 0 \to C = 100$. Plugging in $t = 1$ hour $\to k = \ln\frac{3}{4}$. Therefore the desired equation is: $10 = 100\left(\frac{3}{4}\right)^t \to \log_{3/4} 1/10$. **A**

10. Simplify follow the instructions and add a constant of proportionality. $\frac{dA}{dt} = k(M - A(t))$.
C

11. Separating this differential allows for an easy solve. $\frac{dA}{M-A(t)} = kdt \to \int \frac{dA}{M-A(t)} = \int kdt \to -\ln\big(M - A(t)\big) = kt + C \to A(t) = M - Ce^{-kt}$. Note since k is the same constant as problem 10, this cannot simplified to $A(t) = M + Ce^{kt}$. **A**

12. Given are the following pieces of information $A(0) = 0$, $A(100) = 50$, and $M = 100$. Given this, $A(t) = 100(1 - e^{-\frac{t}{100}\ln 2})$. We hence need to solve $75 = 100(1 - e^{-\frac{t}{100}\ln 2})$. $t = 200$ or 100 more hours. **B**

13. Polynomial equations of the form $f(x) = x^d$ are the form such that this differential equation is true for any value of c. $f'\left(\frac{a}{x}\right) = a^{d-1} * dx^{1-d}$ and $\frac{x}{f(x)} = x^{1-d}$. Hence, values of a will make the following equal to each other. **D**

14. d_n takes the form: $d_n = \det\begin{pmatrix} \cos 1 & \cos 2 & \cos 3 & \cdots & \cos n \\ \cos(n+1) & \cos(n+2) & \cos(n+3) & \cdots & \cos(2n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}^T$. We can prove that $d_n = 0$ by showing that the sum of two rows is in fact a scalar multiple of another row in the determinant. The sum of the first and the third rows is the row: $= \begin{pmatrix} 2\cos(2)\cos(-1) \\ 2\cos(n+2)\cos(-1) \\ \vdots \end{pmatrix}$. This row is the same as row two, except with a scalar multiple of $2\cos(-1)$. Hence, $d_n = 0$. **B**

15. $(fgh)' = f'(gh) + fg'h + fgh' = \frac{(fg)'h + (gh)'f + (hf)'g}{2}$. Evaluating at 0 we have: $\frac{6*3+4*1+5*2}{2} = 16$. **D**

16. Factoring the right side to be able to separate and solve: $(1+y)+x^2(1+y) = (1+x^2)(1+y) \rightarrow \frac{dy}{1+y} = (1+x^2)dx \rightarrow \int \frac{dy}{1+y} = \int (1+x^2)dx \rightarrow y = Ce^{x+\frac{x^3}{3}} - 1$. **A**

17. The question asks for all solutions to the differential equation. All of the given answer choices do not have a constant of integration, therefore they cannot be correct (the answer is choice A + a constant of integration). **E**

18. Consider the answer choices. It appears that x needs to be independent of the differential equation so A is eliminated. Plotting points reveals that $\frac{dy}{dx} = 2 - y$ is the best fit for the slope field present. **B**

19. Process of elimination is another excellent approach for this slope field question. It can be seen to be the correct answer by taking the derivative of equation. $y' = \frac{x^3}{y^2}$. The variable that changes signs of the slope is x which matches that on the slope field. **A**

20. We need to check which time the velocity is 0 and check if the sign changes at these points. $s' = v = 3t^2 - 12t + 9 = 0 \rightarrow t = 1,3$. Since this is quadratic, they change signs at both times. **C**

21. In order to get to 1.2, we need to do two cycles of Euler's method. $y(1.1) = y(1) + \Delta x\left(y'(1)\right) \rightarrow y(1.1) = 0 + .1(1) = .1$. $y(1.2) = .1 + .1(1.2) = .22$. **B**

22. The highest derivative in this equation is a third derivative. Hence, 3rd order. **C**

23. It is fair to treat the water as relatively equal throughout the duration of the problem since the rate coming in is the same as the rate going out. A differential equation describing the changing amount of salt inside the tank is equal to the rate coming in minus the rate going out. Rate going in = $\left(2\frac{kg}{L}\right)\left(3\frac{L}{min}\right) = 6\ kg/min$. Rate going out = $-\frac{3}{100}S$. The overall differential equation is: $\frac{dS}{dt} = 6 - \frac{3}{100}S$. Solve this equation either by using a characteristic polynomial or with an integrating factor. The general form of the solution is $200 + c_1e^{-\frac{3t}{100}}$. Plugging in $S(0) = 0, c_1 = -200. S(t) = 200 - 200e^{-\frac{3t}{100}}$. $S(100) = 200 - 200e^{-3}$. **E**

24. $\frac{200\ kg}{100\ L} = 2\ kg/L$. **C**

25. Implicitly derive this two variable expression with respect to x: $6x - 2xy^3 - 3x^2y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{2xy^3-6x}{4-3x^2y^2} \rightarrow \frac{dy}{dx} = -3$. Using differentials: $f(1.97) = 0 + (-3)(-.03) = .09$. **A**

26. An obvious start is to attempt to factor this expression. Difference of cubes yields $\sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right) = (\sin^2\left(\frac{x}{4}\right) + \cos^2(\frac{x}{4}))(\sin^4\left(\frac{x}{4}\right) - \sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right) + \cos^4\left(\frac{x}{4}\right)) = \sin^4\left(\frac{x}{4}\right) - \sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right) + \cos^4\left(\frac{x}{4}\right) = (\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right))\text{^}2 - 3\sin\text{^}2\left(\frac{x}{4}\right)\cos\text{^}2\left(\frac{x}{4}\right) \rightarrow 1 - \frac{3}{4}\sin x \rightarrow f^{2017} = -\frac{3}{4}$. **E**

27. For the purposes of this problem, lets assume that $|x| < 1$ so that we can sum up the summation. $\frac{dy}{dx} = \frac{1}{1+x} \rightarrow y = \ln|1 + x| + C$. Using the initial condition, $y\left(\frac{1}{2}\right) = \ln 3\,, y = \ln(1 + x) + \ln 2$. $y\left(\frac{3}{4}\right) = \ln\left(\frac{7}{2}\right)$. **B**

28. The pattern will be seen if the first few terms are drawn out. $g = x + e^x - \sin x\,,\ g' = 1 + e^x - \cos x\,,\ g'' = e^x + \sin x\,,\ g''' = e^x + \cos x\,, g^{(4)} = e^x - \sin x\,, g^{(5)} = e^x - \cos x$. Hence, it will repeat after 4 derivatives. The function and its first derivative both have values 1. The sum of 4 derivatives is equal to 4 since the trig terms cancel out. Four goes into 2017, 504 times with one left over. Hence the answer is $4(504) + 2 = 2018$. **C**

29. Begin by taking the derivative of both sides of the equation: $\frac{dy}{dx} = \frac{x}{y-1}$. Hence, the tangent line through point (a,b) that follows this requirement has the form: $y = \left(\frac{a}{b-1}\right)(x - a) + b \rightarrow b^2 - a^2 = b$. Another equation from the equation of the curve is $a^2 - (b-1)^2 = 1$. Solving, by elimination results in $a = \pm\sqrt{2}, b = 2$. Hence, the positive root is the only that gives a positive slope. $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. **C**

30. First, take the derivative. $\ln f = x\ln x \rightarrow \frac{f'}{f} = 1 + \ln x \rightarrow f' = x^x(1 + \ln x)$. Now, we can solve the equation: $f'(x) = f(x) \rightarrow x^x = x^x(1 + \ln x) \rightarrow x^x = 0$ or $1 + \ln x = 1$. Hence, the only solution is 1. **B**