ANSWERS

1. C
2. E (12)
3. B
4. E (2)
5. A
6. C
7. C
8. D
9. D
10. D
11. C
12. E (4)
13. A
14. A
15. A
16. C
17. C
18. C
19. A
20. D
21. D
22. A
23. A
24. B
25. C
26. A
27. E (28)
28. D
29. C
30. A

SOLUTIONS

1. Let the base be $b$. Then $(2b+4)^2 = 5b^2 + 5b + 4$, or $4b^2 + 16b + 16 = 5b^2 + 5b + 4$. Rearranging, we have $0 = b^2 - 11b - 12 = (b-12)(b+1)$ so that $b = 12$. We reject the solution of $b = -1$ since the base must be positive integer greater than 1.

2. Using the change-of-base formula gives us $\frac{\log 36}{\log 7} \times \frac{\log 49}{\log 5} \times \frac{\log 25}{\log 6} \times \frac{3}{2} = \frac{2\log 6}{\log 7} \times \frac{2\log 7}{\log 5} \times \frac{2\log 5}{\log 6} \times \frac{3}{2} = 2 \times 2 \times 2 \times \frac{3}{2} = 12$.

3. Combining the terms on the left-hand side gives us

$$\frac{-4x+40}{x-7} = \frac{4x-40}{13-x}.$$

Cross-multiplying and setting the equation equal to zero results in $(x-13)(4x-40) - (x-7)(4x-40) = 0$ whose only solution is $x = 10$.

4. The function $f(x) = -3\pi\sin(3x-3)$ has amplitude $|-3\pi| = 3\pi$, period $P = \frac{2\pi}{3}$, and phase shift $S = \frac{3}{3} = 1$. Then $\frac{9PS}{A} = \frac{9\left(\frac{2\pi}{3}\right)(1)}{3\pi} = \frac{18\pi}{9\pi} = 2$.

5. Since N is a multiple of 15, it is a multiple of 3 and 5. Since the only digits used are 7s and 0s, the units digit of N must be 0 for it to be divisible by 5. Thus, the remaining digits must sum to a multiple of 3 for N to be divisible by 3. Hence, we need at least three 7s. To create the smallest possible number, it is easy to see that we must have $N = 7770$.

6. Setting $f'(x) = \frac{1}{x} - \frac{k}{2}x^{-1/2} = 0$ and letting $x = 4$ gives $\frac{1}{4} - \frac{k}{4} = 0$. Hence, $k = 1$.

7. Let $N$ be the number. Then $\frac{50}{N} - 3\frac{3}{4} = \frac{50}{N+3}$. Multiplying through by $4N(N+3)$ yields $200(N+3) - 15N(N+3) = 200N$. This can be rewritten as $15N^2 + 45N - 600 = 0$ which factors into $15(N+8)(N-5) = 0$. Since $N$ is positive, we must have $N = 5$.

8. The men can plant $\frac{1000}{9}$ trees per day. This is $\frac{1000}{9} \div 30 = \frac{100}{27}$ trees per day per man. So 36 men would plant a rate of $\frac{100}{27} \times 36 = \frac{400}{3}$ trees per day. To plant 4400 trees then takes the 36 men $4400 \div \frac{400}{3} = 4400 \times \frac{3}{400} = 33$ days.

9. Statement A is not true since one would have to check for a sign change of the second derivative. Statement B is not true since the graph of $y = |x|$ is always nonnegative, implying the area under the graph over $[-1, 1]$ cannot be zero. Statement C is not true since concavity does not determine which Riemann sum is larger or smaller; it is whether the graph is increasing or decreasing. Statement D is true.

10. There are 6 pennies, $n$ nickels, and $d$ dimes. Then, when switched, there are $d$ nickels, $n$ pennies, and 6 dimes. The values are the same. Hence, in cents, $6 + 5n + 10d = 5d + n + 60$, or $4n + 5d = 54$. The only solutions to this equation in integers are $(n, d) = (1, 10), (6, 6), (11, 2)$. The greatest possible number of coins is obtained with $(n, d) = (11, 2)$ which gives $6 + 11 + 2 = 19$ coins.

11. First, we solve $p = -\frac{1}{4}x + 120$ for $x$ to get $x = -4(p - 120) = 480 - 4p$. Then $C = \frac{1}{2}\sqrt{x} + 600 = \frac{1}{2}\sqrt{480 - 4p} + 600 = \sqrt{120 - p} + 600$. Hence $a + b = 120 + 600 = 720$.

12. We have $AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & t \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & t+2 \\ 0 & -t-4 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & t \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2-t & -2+2t \\ -3 & -5 \end{bmatrix}$. Hence, $AB - BA = \begin{bmatrix} t-1 & -t+4 \\ -3 & -t+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & -3 \end{bmatrix}$. We require $t - 1 = 3$ so that $t = 4$. Checking, $t = 4$ does indeed give the other values of 0 and $-3$.

13. We rewrite the equation as $\frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} + \cos(\theta) = 0$ then get a common denominator to arrive at $\frac{1+\sin(\theta)+\cos^2(\theta)}{\cos(\theta)} = 0$. Using the Pythagorean Identity, this becomes $\frac{1+\sin(\theta)+1-\sin^2(\theta)}{\cos(\theta)} = 0$, or $\frac{\sin^2(\theta)-\sin(\theta)-2}{\cos(\theta)} = 0$. Factoring the numerator yields $\frac{(\sin(\theta)-2)(\sin(\theta)+1)}{\cos(\theta)} = 0$. Thus, $\sin(\theta) = -1$ and so $\theta = 270°$. However, this angle makes the denominator of the original equation zero, and thus, the expression is undefined. Therefore there are no values which make the equation true.

14. Triangle $JMK$ is isosceles, so $\angle KJM = \angle JKM = x$. Then $\angle JMK = 180° - 2x$, which implies that $\angle JML = 2x$. Since JM is a median, $KM = ML$. Since triangle $JMK$ is isosceles, $JM = KL$. Thus, $JM = KM = ML$ and triangle $JML$ is isosceles as well. Thus, $\angle MJL = \angle JLM = 90° - x$. Now, $\angle KJL = \angle KJM + \angle MJL = x + 90° - x = 90°$ and hence triangle $JKL$ is right with right angle at $J$. Finally, $\tan(x) = \frac{JL}{JK} = \frac{JL}{1} = JL$.

15. Separating variables gives $\sec^2(y)\, dy = \cos(x)\ dx$. Integrating both sides yields $\tan(y) = \sin(x) + C$. Using the initial condition, we have $\tan\left(\frac{\pi}{4}\right) = \sin(0) + C$, so that $C = 1$. Hence the solution is $\tan(y) = \sin(x) + 1$.

16. The quadratic has more than one real root when the discriminant is positive. The discriminant is $4^2 - 4(a-3)(a) = -4a^2 + 12a + 16$. Since this must be positive, we have $-4(a^2 - 3a - 4) = -4(a-4)(a+1) > 0$. The solution interval is $(-1, 4)$, and the integers which lie in this interval are 0, 1, 2, and 3. However, when $a = 3$, the given equation is not quadratic: it is $4x + 3 = 0$ which has only one real root. Thus the sum of the integer values of $a$ which lead to more than one real root is $0 + 1 + 2 = 3$.

17. We have $a^2 + b^2 = 25$ and $\frac{1}{2}ab = 3\frac{9}{25} = \frac{84}{25}$. Thus $2ab = \frac{336}{25}$ and so $a^2 + 2ab + b^2 = 25 + \frac{336}{25} = \frac{961}{25}$. This implies $(a+b)^2 = \frac{961}{25}$, or $a + b = \frac{31}{5}$. Likewise, $(a-b)^2 = 25 - \frac{336}{25} = \frac{289}{25}$ so that $a - b = \frac{17}{5}$. Finally, $a = \frac{24}{5}$ and $b = \frac{7}{5}$ and we obtain $\frac{a}{b} = \frac{24}{7}$.

18. A sum of nine can be attained by rolling 1-2-6 (6 ways), 1-3-5 (6 ways), 1-4-4 (3 ways), 2-2-5 (3 ways), 2-3-4 (6 ways), and 3-3-3 (1 way) for a total of $N = 25$ ways. A sum of ten can be attained by rolling 1-3-6 (6 ways), 1-4-5 (6 ways), 2-2-6 (3 ways), 2-3-5 (6 ways), 2-4-4 (3 ways), and 3-3-4 (3 ways) for a total of $T = 27$ ways. Hence $|T - N| = 27 - 25 = 2$.

19. The derivative of $y = x^4$ is $y' = 4x^3$ so the slope of the normal to this curve is $-\frac{1}{4x^3}$. This must equal the slope of the line, which is $-\frac{1}{32}$. Hence, solving $-\frac{1}{4x^3} = -\frac{1}{32}$ gives $x^3 = 8$, or $x = 2$. When $x = 2$, we have $y = 2^4 = 16$. Thus, $16 = \frac{k-2}{32}$ implies that $k = 16 \times 32 + 2 = 514$.

20. Note that $P(10) = 12{,}134$ and the sum of the digits of 12,134 is 11. This implies that the coefficients of the fourth-degree polynomial are 1, 2, 1, 3, and 4 in that order. Hence $P(x) = x^4 + 2x^3 + x^2 + 3x + 4$ and $P(3) = 81 + 54 + 9 + 9 + 4 = 157$.

21. Subtract the first equation from the second to obtain $(B - A)(A + B - 1) = 24$. Since the sum of these two factors is $2B - 1$ (which is odd), then one factor is odd and the other even. The only choices of factors of 24 are therefore 3, 8 and 1, 24. Hence, we have $(B - A, A + B - 1) = (3, 8), (1, 24)$. The first case gives $A = 3, B = 6, C = -85$ which we rule out since $C$ must be positive. The second case gives $A = 12, B = 13, C = 57$.

22. The summation represents a Riemann sum for the function $f(x) = 2x$ over the interval $[-1, 3]$. The limit of the summation is equivalent to the definite integral

$$\int_{-1}^{3} 2x \; dx = x^2]_{-1}^{3} = 9 - 1 = 8.$$

23. Multiply the numerator and denominator by $e^x$. Then we have

$$\int_0^{(\ln 3)/2} \frac{e^x}{e^{2x} + 1} \, dx = \tan^{-1}(e^x)]_0^{(\ln 3)/2} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

24. Statement I is true since $(x, y) = (0, -2)$ for $t = 1$. Statement II is true because the derivative is $\frac{dy}{dx} = \frac{3t^2-3}{2t-1}$ which does become 3 when $t = 2$. Statement III is false because $x'(1) = 1 \neq 0$. Statement IV is true because $\frac{d^2y}{dx^2} = \frac{6t(2t-1)-2(2t^2-3)}{(2t-1)^3}$ which becomes $\frac{58}{125} > 0$ when $t = 3$. Hence, statements I, II, and IV are true.

25. Letting $x = a$ results in the expression becoming indeterminate, so we use l'Hopital's rule. First we rewrite the expression, then apply the rule, take the limit, and simplify.

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}} = \lim_{x\to a}\frac{(2a^3x-x^4)^{1/2}-a^{5/3}x^{1/3}}{a-a^{1/4}x^{3/4}}$$

$$= \lim_{x\to a}\frac{\frac{1}{2}(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-\frac{1}{3}a^{5/3}x^{-2/3}}{-\frac{3}{4}a^{1/4}x^{-1/4}}$$

$$= \frac{\frac{1}{2}(2a^4-a^4)^{-1/2}(2a^3-4a^3)-\frac{1}{3}a^{5/3}a^{-2/3}}{-\frac{3}{4}a^{1/4}a^{-1/4}} = \frac{\frac{1}{2}(a^4)^{-1/2}(-2a^3)-\frac{1}{3}a}{-\frac{3}{4}}$$

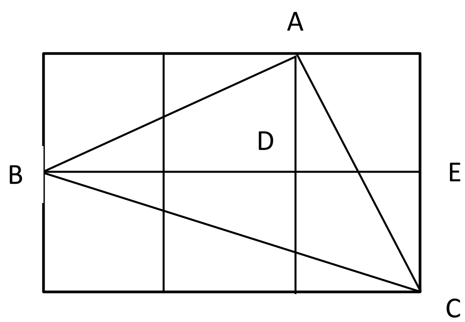
$$= \frac{-a-\frac{1}{3}a}{-\frac{3}{4}} = \frac{\frac{4}{3}a}{\frac{3}{4}} = \frac{16a}{9}$$

(*Note*: This was the first example l'Hopital himself used in his 1696 book on calculus to illustrate the rule which now bears his name.)

26. Note that $f(12) = 12^{2017} + 12 - 13^{2017} + \cos\left(\frac{12}{2017}\right) < 0$ and that $f(13) = 13 + \cos\left(\frac{13}{2017}\right) > 0$. Thus, by the Intermediate Value Theorem, there exists a value $c$ in the interval $(12, 13)$ such that $f(c) = 0$. Hence the value of $k$ is 12.

27. Add $16x^2$ to both sides of the equation to get $x^4 + 2x^3 + 3x^2 + 2x + 1 = 16x^2$. This can be factored into $(x^2 + x + 1)^2 = (4x)^2$ so that $x^2 + x + 1 = \pm 4x$. This leads to the two equations $x^2 - 3x + 1 = 0$ and $x^2 + 5x + 1 = 0$. By the quadratic formula, we arrive at the two pairs of solutions $\frac{3\pm\sqrt{5}}{2}$ and $\frac{-5\pm\sqrt{21}}{2}$. Therefore $a + b + c + u + v + w = 3 + 5 + 2 - 5 + 21 + 2 = 28$.

28. Consider the diagram below consisting of six squares in the shape of a rectangle. Assume the squares are of unit length. Then triangle $ADB$ is right with legs of lengths 1 and 2 and hypotenuse $\sqrt{5}$ and triangle $CEB$ is right with legs 1 and 3 and hypotenuse $\sqrt{10}$. Also, the length of $AC$ is $\sqrt{5}$ which implies that triangle $ABC$ is isosceles. Moreover, since $\sqrt{5}^2 + \sqrt{5}^2 = \sqrt{10}^2$, triangle $ABC$ is also right; hence, the measure of angle $ABC$ is $\frac{\pi}{4}$. Now, $\csc^{-1}\sqrt{5} + \cot^{-1} 3 = \sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1}\frac{1}{3} = m\angle ABD + m\angle CBD = m\angle ABC = \frac{\pi}{4}$.




29. Let $f(n)$ denote the maximum penalty points obtained for an initial interval of length $n$. Clearly, $f(n)$ is nondecreasing, and that $f(1) = 0$, $f(2) = 1$, and $f(3) = 2$. Asking a comparison question, we divide the given interval into two smaller ones; after the answer we can choose one of them; if we choose the right one, we get a one point penalty and if we choose the left one we get a two point penalty. Let $a$ and $b$ denote the lengths of the left and right intervals. Then $f(n) \geq 1 + \max(1 + f(a), f(b))$, with equality exactly when we have divided the given interval in the "best" way. Because the function $f$ is nondecreasing, in the best case we have $a \leq b$. A sufficient condition for the partition to be the best is $1 + f(a) = f(b)$. Suppose there exists such $a$ and $b$, where $a < b$, and set $a_0 = a_1 + a_2$, $a_1 = b$, $a_2 = a$, $a_3 = a_1 - a_2$. Then $a_0 = a_1 + a_2$ and $a_1 = a_2 + a_3$. Also, $f(a_0) = 1 + f(a_1)$ and $f(a_1) = 1 + f(a_2)$. If we suppose again that $f(a_2) = 1 + f(a_3)$, then we can continue by putting $a_4 = a_2 - a_3$ so that $a_2 = a_3 + a_4$, and so on, until we obtain for some $k$ the final answer where $a_k = a_{k+1} = 1$. In reverse order, this process gives the beginning the Fibonacci numbers. Thus all our assumptions about which will be best are true if our given number $n$ is a Fibonacci number, which in our problem, it is. Consequently, by diving the given interval 55 into intervals of lengths $55 = 34 + 21$, $34 = 21 + 13$, $21 = 13 + 8$, $13 = 8 + 5$, $8 = 5 + 3$, $5 = 3 + 2$, $3 = 2 + 1$, $2 = 1 + 1$, we conclude that the problem will be solved with a maximum of 8 penalty points.

30. Suppose the teams are placed into two divisions. Team #3 will make it to the final round if and only if Team #1 and Team #2 are in the opposing division from Team #3. Choose a division for Team #1. Then there are 63 possible placements out of 127 for Team #2 to be in Team #1's division. There are then 64 possible placements out of 126 for Team #3 to be in the opposing division. Hence, the probability Team #3 will make it to the final round is $\frac{63}{127} \times \frac{64}{126} = \frac{32}{127}$.