Answers:

1. E
2. C
3. B
4. C
5. E
6. A
7. B
8. C
9. B
10. A
11. B
12. E
13. A
14. A
15. D
16. B
17. A
18. E
19. A
20. D
21. D
22. A
23. C
24. D
25. D
26. E
27. B
28. B
29. C
30. C

Solutions:

1. The first step to this problem is to establish the form of the gradient vector. Setting this whole ellipsoid equal to a function F we can take the gradient for the normal vector. $\vec{n} = \nabla F = <2x, 8y, 18z>$. Let's allow all points of the form (a, b, c) to be on the ellipse and have normal lines through the origin. Thus, the line containing the normal vector is of the form: $x = a(2t+1), y = b(8t+1)$, and $z = c(18t+1)$. Since this must travel through the origin, we can set each component equal to 0 and solve. In the case of x, either $a = 0$ or $t = -\frac{1}{2}$. In the case of y, either $b = 0$ or $t = -\frac{1}{8}$. In the case of z, either $c = 0$ or $t = -\frac{1}{18}$. Going case by case, if $t = -\frac{1}{2}$ then b must be 0 and c must be 0. Hence, $a = \pm 4$. Likewise, the same is true for the other values of t found. Therefore, there are 6 solutions possible. **E**

2. The easiest method is to convert to cylindrical coordinates. Our functions become $z = 9 - r^2$, and $r = 2$. Therefore, our bounds become: $z = 0$ to $z = 9 - r^2, r = 0$ to $r = 2$, and $\theta = 0$ to $\theta = 2\pi$. Hence the desired triple integral is: $\int_0^{2\pi} \int_0^2 \int_0^{9-r^2} r dz dr d\theta = 28\pi$. **C**

3. The boundaries given in the integral provided suggest a triangular region to integrate over in the xy-plane. Solving for x in all of the limits of integration suggest that our new integral must be of the form: $\int_0^2 \int_0^{2y} f(x,y) dx dy$. Consequently, $A = 0, B = 2, C = 0, D = 2$. The desired sum is thus 4. **B**

4. The following limit can be most efficiently interpreted by converting to polar coordinates. Making the substitutions $x = r\sin\theta$ and $y = r\sin\theta$, the limit can be reduced to $\lim_{r\to 0} r \frac{(\sin\theta)^4 + (\cos\theta)^4}{((\cos\theta)^2 + (\sin\theta)^2)^{\frac{3}{2}}} = 0$. **C**

5. The best way to understand the region of interest is to attempt to draw the boundary region. The points can be plotted in an analogous pattern of polar coordinates and is a cardioid in the xz-plane. And since θ is a free variable it extends from 0 to 2π. ρ extends from the origin at $\rho = 0$ to $\rho = 1 - \cos\theta$. Lastly, since the shape is 3-dimensional ϕ extends form $\phi = 0$ to $\phi = \pi$. Hence the integral is: $\int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\theta} \rho^2 \sin\phi d\rho d\phi d\theta$ and the sum is closest to 9. **E**

6. Simplifying the expression results in the simpler expression:

$\left(f_{xy}(0,0) - f_{yx}(0,0)\right)^2$. Since $f_{xy} = f_{yx}$ for all continuous and differentiable functions of x and y, this is equal to 0. **A**

7. Take both partial derivatives. $z_x = 2x + 2y - 6$ and $z_y = 2x + 4y + 8$. Setting both of these equations equal to zero we obtain two equations with one solution $(10, -7, -58)$. The desired sum is 3. **B**

8. A direction derivative can be used to find the rate of change in a particular direction. Since we are looking for the maximum change, the gradient vector is this maximum. Hence the gradient of the particular function is: $\nabla T = < yz, xz, xy >$. At this point the vector is: $\nabla T =< 4,3,12 >$ and the unit vector is $\nabla T =< \frac{4}{13}, \frac{3}{13}, \frac{12}{13} >$. The sum is $19/13$. **C**

9. To fit the geometric description of the cross product, the two vectors given must be perpendicular to another vector in the direction of $\vec{v}$. Hence, $< 1,1,-5 > \times < 3,-2,0 > = < -10,-15,-5 >$. A vector in this direction that satisfies the third condition is $\pm < 2,3,1 >$. The desired sum of absolute value of the coordinates is 6. **B**

10. The best way to solve this problem is to write out the chain rule expression. $f_u = f_x x_u + f_y y_u + f_z z_u$. Plugging in for what is given we have: $2 = 0 + f_y + 5$. $f_y = -3$. **A**

11. In order to better understand this region we can split up the inequality. The first half of the inequality is equivalent to $z \geq 1$ in rectangular coordinates. The second half of the inequality is equivalent to $x^2 + y^2 + (z-1)^2 \leq 1$. Hence, the region of interest is the top half of this sphere. Consequently, ρ extends from $\rho = \sec\phi$ to $\rho = 2\cos\phi$. ϕ extends form $\phi = 0$ at the top of the shape, to $\phi = \frac{\pi}{4}$ which is where the two intersect (this can be most efficiently seen by seeing the 45-45-90 triangle formed at the intersection). Lastly, θ extends all the way around from $\theta = 0$ to $\theta = 2\pi$. The integral becomes: $\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta$. Hence the sum is closest to 10. **B**

12. Visualizing the same picture that was understood in problem 11, it can be seen that we are again taking the volume of an elevated top half of a sphere. Hence, the desired integral can be found to be $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1}^{1+\sqrt{1-x^2-y^2}} dzdydx$. The sum of all the limits of integration is $2 + \sqrt{1-x^2-y^2}$. The maximum of this elevated half sphere is 3. **E**

13. Since the region of interest is complicated, we could attempt to prove surface independence. Taking the divergence of the vector field we see that it is surface independent: $\vec{\nabla} \cdot \vec{F} = 0 - x + x = 0$. Now since the flux is independent of the region we can see that now we want to integrate over the downwards oriented ellipse (E). $\iint_S \vec{F} \cdot d\vec{S} = \iint_E \vec{F} \cdot d\vec{S} =$

$\iint_E \vec{F} \cdot \vec{n} dS = \iint_E < y^2z - z^2, 4 - xy, 3 + xz > \cdot < 0,0,1 > dS = -\iint_E (3 + xz) dS =$
$-3(area\ of\ ellipse) = -18\pi$. **A**

14. We can use the vectors perpendicular to the two planes $< 2,3,-1 >$ and $< 4,5,1 >$. The angle can be found by $\cos\theta = \frac{<2,3,-1>\cdot<4,5,1>}{\sqrt{14}\cdot\sqrt{42}} => \theta = \cos^{-1}(\frac{11}{21}\sqrt{3})$. **A**

15. Applying Green's Theorem to the following line integral yields the following simplification $\int \int_R (3-7) dA = -4(Area\ of\ Triangle) = -4\left(\frac{1}{2}\right)(4) = -8$. But since the direction that we are traveling is clockwise, we need to take the negative of this area. Hence, the line integral simplifies to 8. **D**

16. $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x => x^3 = y$ and $y^3 = x$. Thus, $x = \pm y$ and critical points exist at $x = 1, x = -1$, and $x = 0$. $x = 0$ is the only saddle using the second derivative test. **B**

17. div $\vec{F} = \nabla \cdot F = 0$. Likewise, curl $\vec{F} = \nabla \times F = \langle 2y - 2z, 2z - 2x, 2x - 2y \rangle$. At the point given this curl is equal to 0. Hence, the divergence plus the curl at the desired point is 0. **A**

18. This problem is easiest solved by first directly solving for R. $R = \frac{R_1R_2}{R_1+R_2}$. Multivariable differentials will allow us to approximate error in this case. $dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2$. $\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1+R_2)^2}$. Plugging in gives $\frac{4}{9}$. $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}$. Likewise, substitution leads to $\frac{1}{9}$. Plugging into the dR equation gives the maximum error. $dR = \left(\frac{4}{9}\right)(3) + \left(\frac{1}{9}\right)(6) = 2$. **E**

19. The best way to approach this limit is to approach the point $(0,0)$ from the line $y = mx$. Plugging in shows: $\lim_{(x,y)\to(0,0)} \frac{x^2(mx)^2}{x^4+(mx)^4} = \lim_{(x,y)\to(0,0)} \frac{m^2}{1+m^4}$. Hence, the limit is different for different values of m. Thus, the limit does not exist. **A**

20. This is a change of variables problem. Let $u = y - x^2$ and $v = y + 2x$ and we are going to integrate over $-1 \le u \le 1$ and $1 \le v \le 3$. Thus, the following change of variables Jacobian that needs to be evaluated is: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2x & 1 \\ 2 & 1 \end{vmatrix} = -2(x+1)$. Hence, the new integral becomes: $\frac{1}{2}\int_1^3 \int_{-2}^1 (u+v) du dv = \frac{9}{2}$. **D**

21. Since this region is a closed and piecewise smooth surface, the divergence theorem can be used. $\nabla \cdot \vec{F} = 5(x^2 + y^2 + z^2)$. Hence, the surface integral can become $\int\int\int_R 5(x^2 + y^2 + z^2)dV$. Solving in cylindrical coordinates is easiest. $= \int_0^{2\pi}\int_0^2\int_0^3 5(r^2 + z^2)rdzdrd\theta = 300\pi$. **D**

22. Given the points the normal vector of the plane can be found by taking the cross product of vectors formed from the points. Two vectors formed by these points are $< -2, -1, -1 >$ and $< 5, 2, -4 >$. Taking the cross product, $\det\begin{pmatrix} i & j & k \\ -2 & -1 & -1 \\ 5 & 2 & -4 \end{pmatrix} =< 6, -13, 1 >$. Hence the plane is of the form $6(x-1) - 13(y-3) + (z-5) = 0 => 6x + 3y + z = -28$. **A**

23. This is Fubini's theorem. **C**

24. $W = \oint_C \vec{F} \cdot \hat{T} ds = W = \oint_C \vec{F} \cdot dr$. Hence this integral can be simplified to: $W = \int_C < -y, x > \cdot < dx, dy > = \int_C -ydx + xdy$. Using the substitution $x = \cos\theta$ and $y = \sin\theta$, this integral becomes: $\int_0^{2\pi} d\theta = 2\pi$. **D**

25. The shape of interest is best understood in the cylindrical coordinate system. The z-coordinate extends from $z = x^2 + y^2 = r^2$ to $z = 4 - 3x^2 - 3y^2 = 4 - r^2$. Theta goes all the way around the unit circle $\theta = 0$ to $\theta = 2\pi$. r extends from $r = 0$ to $r = 1$. The triple integral desired is $\int_0^{2\pi}\int_0^1\int_{r^2}^{4-3r^2} rdzdrd\theta = 2\pi$. **D**

26. This is an application of the multivariable chain rule. The problem tells us that $\frac{\partial F}{\partial x} = 1, \frac{\partial F}{\partial y} = 2, \frac{\partial F}{\partial z} = -2$. Hence $\frac{\partial F}{\partial \phi} = F_x\frac{\partial x}{\partial \phi} + F_y\frac{\partial y}{\partial \phi} + F_z\frac{\partial z}{\partial \phi}$. $\frac{\partial x}{\partial \phi} = \rho\cos\phi\cos\theta = 1$. $\frac{\partial y}{\partial \phi} = \rho\cos\phi\sin\theta = -1$. $\frac{\partial z}{\partial \phi} = -\rho\sin\phi = -\sqrt{2}$. $\frac{\partial F}{\partial \phi} = 1 - 2 + 2\sqrt{2} = 2\sqrt{2} - 1$. **E**

27. The order of integration must be switched in order to evaluate the integral. Hence, sketching the diagram reveals that x now extends from $x = 0$ to $x = \sqrt{y}$ and y now extends from $y = 0$ to $y = 9$. The new, but equivalent integral is: $\int_0^9\int_0^{\sqrt{y}} xe^{-y^2}dxdy = -\frac{1}{4}(1 - e^{-81})$ upon evaluation. **B**

28. Consider the property of double integrals: $\int\int f(x)g(y)dxdy = \int f(x)dx\int g(y)dy$. Hence, the double integral given in the hint can become $\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dydx =$

$\int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = X^2$. Converting this integral to polar, we can see that it can be solved. $\int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dydx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta \,.= \frac{\pi}{4}$. $X^2 = \frac{\pi}{4} => X = \frac{\sqrt{\pi}}{2}$. **B**

29. The plane given in the problem intersects the xy-plane on the line: $3x + 2y = 6$. This line can be written in parametric form as follows: $x = t, y = 3 - \frac{3}{2}t, z = 0$. Hence, a vector in this direction is: $v =< 1, -\frac{3}{2}, 0 >$ and a point on this plane is: (0,3,0). Forming a new vector also on the plane of interest with the points $(0,3,0)$ and $(1,1,1)$ gives $u =< 1, -2, 1 >$. Cross product $u \times v$ gives the normal vector to the plane $< -\frac{3}{2}, -1, -\frac{1}{2} >$. The plane of interest can now be determined as: $-\frac{3}{2}(x-1) - (y-1) - \frac{1}{2}(z-1) = 0 => 3x + 2y + z = 6$. Sum = 12. **C**

30. The given vector field $\vec{F}$ is independent of path since $\vec{\nabla} \times \vec{F} = \vec{0}$. Consequently, the parameterized curve is useless because we can use the fundamental theorem of line integrals. Integrating the various components we can find a potential function f such that $\nabla f = \vec{F}$. Integration yields the various forms of f. $f = e^{xy} + xz + C(y,z), f = e^{xy} + C(x,z), f = xz + C(x,y)$. Hence comparing the three forms the potential function is: $f = e^{xy} + xz$. Evaluating this function from $t = 0$ to $t = 1$ results in a final answer of $e^{-2} + 1$. **C**