"For all questions, answer choice "E. NOTA" means none of the above answers is correct."

Unless otherwise indicated, you may assume standard notation for a triangle ABC in which capital letters represent angles and lowercase letters represent sides and angle A is opposite side a, angle B is opposite side b, and angle C is opposite side c. For right triangles, C is always 90 degrees and c is always the hypotenuse.

1. Right triangle ABC has $A = \tan^{-1}\frac{7}{24}$ and $a = 14$. Find c.

A. 25 B. 50 C. 625 D. 1250 E. NOTA

Answer B. The information implies that the ration of a:b is 7:24, therefore b=48. After this you can use the Pythagorean Theorem (or recognize that this is a double of a Pythagorean triple 7-24-25) and find that c=50

2. Right triangle ABC has sides such that b and c are two consecutive integers and a=13. What is b+c?

A. 29 B. 39 C. 149 D. 159 E. NOTA

Answer E. Since b and c are consecutive integers, $c - b = 1$. Now, $a^2 = c^2 - b^2 = (c - b)(c + b) = c + b = 169$

3. If triangle ABC has a = 4 and B = 60°, for which value of b can a second triangle with these same three measurements (but not congruent to ABC) be formed?

A. 2 B. $2\sqrt{3}$ C. 3 D. 4 E. NOTA

Answer C. Let side c be a horizontal base with angle B and side a on the left side. If you picture the other end of side a as a hinge for side b, then side b will just barely reach side c when oriented vertically at length $2\sqrt{3}$. Once it is longer than this, but shorter than side A, it can connect to side c at two points, forming two different triangles.

4. If triangle ABC has a = 4 and B = 60°, which value of b is NOT possible?

A. 2 B. $2\sqrt{3}$ C. 3 D. 4 E. NOTA

Answer A. See above.

5. Which of the following is NOT equivalent to the other three? (If all are equal, choose E)

A. $\sin^{-1}\frac{7}{41}$ B. $\cos^{-1}\frac{40}{41}$ C. $\cot^{-1}\frac{40}{7}$ D. $\csc^{-1}\frac{41}{40}$ E. NOTA

Answer D. All of these are one of the acute angles in a 9-40-41 right triangle (or some similar triangle). Let A be the angle from answer A. That implies that the opposite side is 9 and the hypotenuse is 41 and the adjacent is therefore 40. Given these the only one that does not represent that angle is D, since it implies that its opposite side is 40 (contrary to the above).

6. A right triangle can be used to prove the following property of an ellipse: $c^2 = a^2 - b^2$. Which of the following points (relative to the ellipse) is NOT a vertex of that right triangle? (If more than one is not a vertex of the triangle, choose E.)

A. co-vertex B. focus C. center D. vertex E. NOTA

Answer D. Answers A, B, and C are all necessary to produce the quantities b and c in the legs of the triangle, but a is the hypotenuse which is derived not from the fact that it is the distance from the center to the vertex (since the vertex is not used in this triangle), but from the fact that the sum of the distances between the foci and a fixed point is always 2a and that this hypotenuse is the distance from the one focus to the co-vertex (which is symmetrical to the line drawn from the same co-vertex to the other focus).

7. If a tangent line is drawn to a parabola at a point other than the vertex, let the point where the tangent line and axis intersect be A, the point of tangency be B, and the focus be C. What kind of triangle must ABC be (be as specific as possible)?

A. equilateral B. isosceles C. right isosceles D. scalene E. NOTA

Answer B. It is a property of tangent lines to parabolas that they form the base of an isosceles triangle with the focus as the opposite vertex. However, we cannot specify that the triangle will be acute, right, or obtuse since that depends on the point of tangency.

8. Given a triangle ABC, which of the following is NOT equivalent to the other three? (If all are equal, choose E)

A. $a^2 + b^2 - 2ab\cos C$ B. $a^2 - b^2 - 2ac\cos C$ C. $b^2 - a^2 - 2bc\cos C$
D. $\frac{\sin^2 C}{[a\sin(A)]^2}$ E. NOTA

Answer D. Answers A-C are all manipulations of the Law of Cosines equivalent to c^2, but D is a mistaken manipulation of the Law of Sines since a is on the denominator rather than the numerator.

9. In triangle ABC, $a = 67$ and $b = 99$. Which of the following is NOT a possible value for c?

A. 33 B. 83 C. 133 D. 183 E. NOTA

Answer D. The Triangle Inequality states that the sum of any two sides must be larger than the third side. All the answers satisfy this except D since a+b would be less than c in that case.

10. An equilateral triangle is divided into 4 smaller triangles by its 3 mid-segments. One of these triangles is shaded black and the other three are each further subdivided in the same way. One of the smallest new triangles produced this way is shaded black and all others are again subdivided. If this pattern continues infinitely many times, how much of the triangle will be shaded black?

A. $\frac{1}{3}$ B. $\frac{2}{5}$ C. $\frac{7}{16}$ D. $\frac{1}{2}$ E. NOTA

Answer E. The shaded amount forms a geometric series of $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots = \frac{\frac{1}{4}}{1-\frac{3}{4}} = 1.$

Use the following for questions 11-13: ΔXYZ has XY = 6, m<ZXY = 45, and XY ⊥ YZ. Points A(2,1) and B(2,7) are on ΔABC, which lies entirely in the first quadrant, and ΔABC ≡ ΔXYZ.

11. What are the coordinates of C?

A. (-6,7) B. (5,4) C. (8,1) D. (8,7) E. NOTA

Answer D. XYZ is a right isosceles triangle. By CPCTC, $Y = B = 90°$ and $YZ = BC = 6$. This leaves only answers A and D as possibilities, but ABC is entirely in the 1st quadrant, hence D.

12. What are the coordinates of the centroid of ABC?

A. (4, 4) B. (4,5) C. (5,4) D. (5,5) E. NOTA

Answer B. First, we need to find the medians (since the intersection of two of them will give us the centroid). The median through point C has the equation $y = \frac{1}{2}x + 3$. The median through point A has the equation $y = 2x - 3$. Therefore, the centroid is (4, 5).

13. What is the radius of the circle that circumscribes ABC?

A. $2\sqrt{3}$ B. $3\sqrt{2}$ C. $2\sqrt{5}$ D. $3\sqrt{3}$ E. NOTA

Answer B. To find the circumcenter we will use the vertical and horizontal perpendicular bisectors of ABC and get (5,4) as the point of intersection. The distance between this point and any of the vertices is $3\sqrt{2}$.

Use the following information for questions 14-16: Triangle DEF has an inscribed circle and a circumscribed circle with the same center C. The circumscribed circle has circumference 24π.

14. What is m<DCE + m<CFE?

A. 60 B. 90 C. 120 D. 150 E. NOTA

Answer D. DCE is one of three congruent central angles of the circle and therefore 120. CFE is one of two congruent base angles in the obtuse isosceles triangle with 120 as the vertex angle and therefore it is 30. This gives us D for the sum.

15. What is the altitude of DEF?

A. 12 B. 16 C. 18 D. 24 E. NOTA

Answer C. The given information implies DEF is equilateral. Therefore all points of concurrency are the same so the circumcenter is the centroid. The distance from the centroid to a vertex is twice that of the centroid to the opposite side. The former is 12, therefore the latter is 6 giving us a full length of 18.

16. What is the perimeter of DEF?

A. $9\sqrt{3}$ B. $12\sqrt{3}$ C. $27\sqrt{3}$ D. $36\sqrt{3}$ E. NOTA

Answer D. Given the above, the side length of DEF must be $12\sqrt{3}$. Multiplying by 3 since DEF is equilateral gives answer D.

17. What are the angles of the triangle made by the y-axis and the lines $y = \sqrt{3}x-5$ and $y = -\sqrt{3}x+3$?

A. 30-30-120 B. 30-60-90 C. 45-45-90 D. 60-60-60 E. NOTA

Answer A. The angle made between each of the lines and the y-axis can be found most easily by constructing a small right triangle with a hypotenuse of 2 along the line. Since the slope of the lines are $\pm\sqrt{3}$, the legs of such triangles would be 1 (horizontally) and $\sqrt{3}$, (vertically) making them 30-60-90 triangles. This means the lines form 30 degree angles with the y-axis and therefore A is the correct answer.

18. What are the coordinates of the other vertex (in the first quadrant) of an equilateral triangle with one vertex at the origin and another at (0, 4)?

A. $(\frac{2\sqrt{3}}{3}, 2)$ B. $(\sqrt{3}, 2)$ C. (3, 2) D. $(2\sqrt{3}, 2)$ E. NOTA

Answer D. The altitude of the triangle would have endpoints at (0,2) and the vertex in question, forming two 30-60-90 triangles with a short leg of 2. Therefore, the longer leg must be $\sqrt{3}$ times that which gives us answer D.

19. A farmer wants to build a fence surrounding a portion of field in the shape of a right triangle with one of its sides bordered by a river (so that no fencing is required on that side). If he only has 100 feet of fence what is the largest such area he can enclose (in $ft.^2$)?

A. 1152 B. 1250 C. 1825 D. 2500 E. NOTA

Answer B. The side along the river should be the hypotenuse since it will allow him to get the most out of his 100 ft. of fence. If he uses x feet along one of the legs the other leg must necessarily be 100-x feet in length. Using these we get the area equation $A = \frac{100x-x^2}{2}$. Since the equation is a downward facing parabola the maximum value will occur at the vertex, whose x-coordinate is given by $-\frac{b}{2a} = 50$. This is one leg so the other is also 50 yielding answer B.

20. If triangle ABC has an area of 36, a perimeter of 18, and a side of length 8, what is the product of the other two side lengths?

A. 9 B. 16 C. 21 D. 25 E. NOTA

Answer C. Let x be one unknown side length and the other will be 10-x. Heron's formula can be used to create the equation $6\sqrt{3} = \sqrt{9(1)(9-x)(x-1)}$ which can be simplified to $-12 = x^2 - 10x + 9$, whose solutions are 3 and 7, representing each side (remember that the other side is 10-x). Therefore C is the answer.

21. If an acute triangle has sides of length 9 and 12 and an area of $27\sqrt{3}$, what is the measure if the angle (in degrees) formed by the two given sides?

A. 30 B. 45 C. 60 D. 90 E. NOTA

Answer C. The area of a triangle can be given by $A = \frac{1}{2}ab\sin C$. Since we have A, a, and b, we simply need to solve for C which is 60 degrees.

22. Which of the following is NOT is a triangle congruence shortcut?

A. AAS B. ASA C. SSA D. SAS E. NOTA

Answer C.

23. A triangle has vertices at (0, 6), (3, 0) and the origin. What is the area of the largest rectangle (with one vertex at the origin) inscribed in such a triangle?

A. 1.5 B. 3 C. 4.5 D. 5 E. NOTA

Answer C. The line forming the hypotenuse is $y = -2x + 6$. If the width of the rectangle is x, then the height is $-2x + 6$, giving us an area of $A = -2x^2 + 6x$. This is another downward facing parabola so the vertex will give us the maximum area. The vertex is at 1.5 which would be the width of the rectangle, giving it a height of 3, giving us answer C.

24. In Square ABCD, let E and F be the midpoints of two opposite sides and P be the midpoint of EF. If a line was drawn from point P to each of the other six points mentioned, what are the angle measures of the smallest triangles (by area) formed by doing this?

A. 20-70-90 B. 30-60-90 C. 40-50-90 D. 45-45-90 E. NOTA

Answer D. This process forms six small triangles within the square (without counting compositions of smaller triangles into larger ones, e.g. ABC). This is true regardless of the position of E and F by symmetry. Each and every one of these triangles, however, is a 45-45-90 triangle. This is perhaps most easily seen by drawing one more line between the other two midpoints of the sides of the square (not E and F) so that you now have 8 congruent triangles in the square. Since there are 8 angles whose vertex is at P we can divide 360 by 8 and get 45 for the measure of each of those. Since each of those 8 triangles are also right triangles, that makes them all 45-45-90 triangles.

25. Which of the following expresses the area of an isosceles right triangle in terms of its perimeter?

A. $\frac{P^2(3-2\sqrt{2})}{2}$ B. $\frac{P^2(3+2\sqrt{2})}{2}$ C. $P^2(3-2\sqrt{2})$ D. $P^2(3+2\sqrt{2})$ E. NOTA

Answer E. Let s be the leg of the triangle. The perimeter would be $(2+\sqrt{2})s$, therefore, s can be expressed in terms of p by $\frac{p}{2+\sqrt{2}} = \frac{p(2-\sqrt{2})}{2}$. The area in terms of s would be $\frac{1}{2}s^2 = \frac{1}{2}\left[\frac{p(2-\sqrt{2})}{2}\right]^2 = \frac{1}{8}p^2(6-4\sqrt{2}) = \frac{1}{4}p^2(3-2\sqrt{2})$

26. Find $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}$, in degrees.

A. 90 B. 180 C. $\sin^{-1}\frac{3}{4}$ D. $\sin^{-1}\frac{7}{5}$ E. NOTA

Answer A. The two terms in this expression are representative of the two acute angles in a 3-4-5 right triangle, therefore their sum must be 90.

27. Given the following system, find $\frac{5bc}{26}$ (all variables are positive):

$$a^2+b^2=c^2$$
$$b^2=c^2+a^2-\frac{10ac}{13}$$
$$a^2=c^2+b^2-\frac{24bc}{13}$$

A. $\frac{25}{2}$ B. 25 C. 30 D. 60 E. NOTA

Answer C. The system represents the Law of Cosines applied 3 times to a 5-12-13 right triangle (the first simplifies to the Pythagorean Theorem). The expression in question is a simplified form of the expression for the area of a triangle: $\frac{1}{2}bc\sin A$. Therefore, the answer is 30, the area of a 5-12-13 right triangle.

28. Line segment AF is subdivided into 5 smaller segments by points B, C, D, and E (in that order) such that AB = EF and BC=DE. If B, C, D, and E were allowed to act as pivots so that the line segments could rotate about them arbitrarily, and they were rotated in such a way as to produce three congruent equilateral triangles then what is the ratio of AB:BC:CD?

A. 1:1:1 B. 1:2:3 C. 2:1:3 D. 3:1:2 E. NOTA

Answer C. Each of the segments is going to form one or more sides of the triangles. Let's first consider the middle segment CD (imagining AF as horizontal). It will form the base of all three triangles since BC and DE will need to pivot upward (or downward) and then AB and EF will need to pivot in the opposite direction crossing through CD. BC and DE will only form one side each of the outer two triangles. AB and EF will form the remaining side of these, but also one side each of the middle triangle (after crossing through CD). This gives us C as the answer.

29. Which of the following represent the side lengths of a right triangle?

A. 37-679-680 B. 39-755-756 C. 41-840-841 D. 43-934-935 E. NOTA

Answer C. There is a property of right triangles that if the short leg is an odd number integer length and the other two sides are consecutive integers then their sum is equal to the square of the short side (this can be easily proved). Using this property the four choices may be checked much more quickly and we can arrive at C as the answer.

30. Triangle XYZ is inscribed in circle C. Arc XY measures 50 degrees and angle YCZ measures 100 degrees. What is the measure of angle XYZ?

A. 30 B. 75 C. 105 D. 120 E. NOTA

Answer C. An inscribed angle is equal to half the measure of the intercepted arc (whereas a central angle is equal in measure to it). This with the given information tells us the two other angles of the triangle, 25 and 50 degrees respectively. Therefore the remaining angle is 105.