Answers:

1. 13
2. 1
3. 175
4. 78
5. $\frac{17}{5}$
6. 5
7. 51
8. Indirect Proof or Proof by Contradiction
9. 3
10. 60π
11. $6\sqrt{6}$
12. 108
13. A plane
14. 1350
15. 81π
16. 15
17. $18 + 18\sqrt{2}$
18. 36
19. $240\sqrt{10}$
20. 90
21. 9
22. 210
23. $\sqrt{\pi}$
24. 84
25. $\frac{25}{49}$

Solutions:

1. We know the perimeter can be expressed as $ns = 144$ where *n* is the number of sides and *s* is the length of each side. Then $s = \frac{144}{n}$ and *s* must be an integer, so the number of sides must be a factor of 144. One can determine that 144 has 15 positive integer factors (1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, and 144), and each of these values for *n* would give an integer number of sides. However, a polygon may not have 1 side or two sides, so only 13 of the factors are acceptable values for *n*.

2. We are looking for lattice points on or inside the circle which are also in Quadrant I. This gives the following constraints: $x \geq 1, y \geq 1$, and $(x+3)^2 + (y+2)^2 \leq 25$. If we start by trying $x = 1$, then the equation for the circle becomes $(1+3)^2 + (y+2)^2 \leq 25$ or $(y+2)^2 \leq 9$, and the only positive integer value for y which is a solution is 1, so we have the point (1, 1). If $x = 2$, then we have $(2+3)^2 + (y+2)^2 \leq 25$ or $(y+2)^2 \leq 0$ which has positive integer solutions for y. Therefore, the only point satisfying the conditions is (1, 1).

3. The sum of the interior angles in a hexagon is 720°. This means the average angle is 120°. Since there are an even number of angles, (I think about them in an increasing list that I don't yet know: ___, _____, _____, _____, _____, _____), the average angle measure will be in the middle of the progression, between the 3rd and 4th angles. If the difference between the consecutive angles is *d*, then the difference between 120° and the 4th angle is $\frac{d}{2}$, then from the 4th to the 5th is *d*, and from the 5th to the 6th is *d*. So from 120°, to the 6th and largest angle is $\frac{d}{2} + d + d = \frac{5d}{2}$, and we know that the 6th angle measure is less than 180° since the hexagon is convex. This means that $120 + \frac{5d}{2} < 180 \Rightarrow \frac{5d}{2} < 60 \Rightarrow d < 24$. The value for *d* must also be an even integer, since the difference between 120° and the 4th angle is $\frac{d}{2}$. Therefore, $d = 22$ and the measure of the 6th and largest angle is $120 + \frac{5}{2}(22) = 120 + 55 = 175$.

4. The number of sides is *n*. The number of diagonals is $\frac{n(n-s)}{2}$. Then $\frac{n(n-3)}{2} = 5n \Rightarrow n(n-3) = 10n \Rightarrow n - 3 = 10 \Rightarrow n = 13$. So the number of sides is 13, the number of diagonals is 65, and the sum of the two is 78.

5. Don't get into too many details too early! (It is helpful to know all of the lines are in a single plane on this problem.) Line *b* is perp. to line *a*. Line *c* is parallel to line *b*, so perp. to line *a*. Line *d* is perp. to line *c*, so parallel to line *a*. Finally, line *e* is parallel to line *d*, so parallel to line *a*. So we need a line passing through (9, 7) that is parallel to line *a*. The equation of this line is $2x - 5y = -17$, and the y-intercept (when x = 0) is $\left(0, \frac{17}{5}\right)$, so $\frac{17}{5}$ is your y-coordinate.

6. There is a possibility of 5 congruent pairs. The simple part is we know they can be similar and have 3 pairs of congruent angles. So, we now know if all three pairs of sides are congruent, then the triangles are congruent, which we don't want to be the case. So, we shoot for exactly 2 pairs of congruent sides. A specific working solution (and there are infinitely many of them) is ΔABC with side lengths $\sqrt{2}$, $2\sqrt{2}$, and 4, then ΔDEF with sides of length $2\sqrt{2}$, 4, and $4\sqrt{2}$. Their sides are in the same ratio, preserving similarity, and exactly 2 pairs of sides have equal length. Here we have a solution with 5 pairs of congruent parts.

7. The Triangle Exterior Angle Theorem states than in a triangle, an exterior angle is equal to the sum of its two remote interior angles. So $(2x + 25) + (7x - 14) = (4x + 76)$, and this equation leads to an x-value of 13, which in turn leads to $m\angle A = 51°, m\angle B = 77°$, and $m\angle C = 52°$, so the smallest interior angle has measure 51°.

8. This type of proof is called an Indirect Proof, or Proof by Contradiction.

9. Consider citizens B, C, and D. If all of them are liars, then these three would be telling the truth, which is a contradiction, and means not all of them can be liars. Therefore, these 3 must be lying, and citizen A must be telling the truth. There are 3 liars in the group.

10. For an equilateral triangle, $A = \frac{s^2\sqrt{3}}{4} = 36\sqrt{3}$, which leads to $s = 12$. Then the inscribed radius is $r_i = \frac{s\sqrt{3}}{6} = \frac{12\sqrt{3}}{6} = 2\sqrt{3}$ and the circumradius must be $r_c = \frac{s\sqrt{3}}{3} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$. The inscribed circle has area 12π and the circumscribed circle has area 48π, which gives us a sum of 60π.

11. Draw the two circles, which will overlap slightly since the distance between the centers is less than the sum of the radii. Draw the common external tangent, the radii to the points of tangency, and the segment between the centers of the circle. Then draw a segment from the center of the smaller circle perpendicular to the larger radius drawn to the point of tangency. This gives a right triangle with hypotenuse of length 15 (the distance between the centers) and with one leg 3 (the difference between the radii). The length of the other leg is the same as the length of the common external tangent, which we find using the Pythagorean Theorem, and we arrive at $6\sqrt{6}$.

12. Think about the decagon being circumscribed by a circle. Each of the sides, which would then be chords of the circle will cut off congruent arcs, since the chords themselves are congruent. The means that each chord cuts off $\frac{360}{10} = 36$ degrees. When you draw the angle,

you can see $\angle DFH$ is an inscribed angle containing 6 of these arcs, and an inscribed angle is half of its intercepted arc. So $m\angle DFH = \frac{1}{2}(6)(36) = 108°$.

13. If you limited yourself to a specific plane, then the set of all points equidistant from two given points is a line (the perpendicular bisector of the segment between the two points). However, since we are not limited to a plane here, the locus of points can be the entire plane which passes through the midpoint of the segment between the two give points, and is orthogonal to the line passing through the two given points.

14. The triangle is a right triangle. The sine of the smallest angle is $\frac{3}{5}$, which will be the ratio of the short leg to the hypotenuse. This means that it is a 3-4-5 Pythagorean Triple. However, we know $3 + 4 + 5 \neq 180$ but $3 + 4 + 5 = 12$ and $12(15) = 180$ so we need to multiply each side by 15. So the sides of this right triangle are 45, 60, and 75. Then $A = \frac{1}{2}ab = \frac{1}{2}(45)(60) = 1350$.

15. Draw the two circles having the same center, then draw the chord of the larger circle so that it is tangent to the smaller circle. Then draw the radius of the smaller circle to the point of tangency, which will be perpendicular to the chord. Finally, draw the radius of the larger circle to one endpoint of the chord. This creates a triangle with one leg equal to the radius (r) of the smaller circle, one leg equal to half of the chord (or 9 units), and the hypotenuse is equal to the radius (R) of the larger circle. So using the Pythagorean Theorem, we end up with the equation $r^2 + 9^2 = R^2$ which can be manipulated to give $81 = R^2 - r^2$. Multiply both sides by π to get $81\pi = (R^2 - r^2)\pi = \pi R^2 - \pi r^2$ which is the difference between the areas of the circle.

16. The angle bisector is drawn from the largest angle, so it intersects the longest side, cutting it into a 27:36 or 3:4 ratio. I say $3x + 4x = 42 \Rightarrow 7x = 42 \Rightarrow x = 6$ so the pieces of the longer side created by the angle bisector will have length $3x = 18$ and $4x = 24$. If the length of the angle bisector is given by the variable *n*, then the smallest triangle has perimeter 27+18+*n* and the larger of the two triangles has perimeter 36+24+*n*. Then $(36 + 24 + n) - (27 + 18 + n) = 15$. Notice this difference can be produced in an easier way. The difference between 27 and 36 is 9. The difference between 18 and 24 is 6. The angle bisector is in both triangles, so the difference is 0. 9 + 6 + 0 = 15.

17. The triangle that will give us the most area for the perimeter is a 45°-45°-90° triangle. It will also give us the least perimeter for a given area. Then let the legs be of length *x*. We know the area will be $A = \frac{1}{2}bh = \frac{1}{2}(x)(x) = \frac{x^2}{2} = 81$. Then $x^2 = 162$ and $x = 9\sqrt{2}$. Each leg will then have length $9\sqrt{2}$ and the hypotenuse will have length $(9\sqrt{2})(\sqrt{2})$ or 18. Then $P = 18 + 18\sqrt{2}$.

18. $V_{sphere} = \frac{4}{3}\pi r^3 = V_{cone} = \frac{1}{3}\pi r^2 h \Longrightarrow \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h$ so $\frac{4}{3}\pi 9^3 = \frac{1}{3}\pi 9^2 h \Longrightarrow$

$4\pi 9^3 = \pi 9^2 h \Longrightarrow 4\pi(9) = \pi(h) \Longrightarrow h = 36.$

19. Let the dimensions of the right rectangular prism be given by $l, w, and\ h$. Without loss of generality, we say $lw = 80, lh = 60, and\ wh = 120$. If we multiply these equations together, we get $(lw)(lh)(wh) = (80)(60)(120) \Longrightarrow (lwh)^2 = (80)(60)(120) \Longrightarrow lwh = \sqrt{(80)(60)(120)} \Longrightarrow V = 240\sqrt{10}.$

20. Draw the figure as described, then draw segments $\overline{DE}$ and $\overline{DB}$. Working in degrees, we can see that $m\angle ABE = 60$ and $m\angle ABD = 45$, which gives $m\angle DBE = 15$. Then we know $m\angle BAE = 60$ and $m\angle BAD = 90$ so $m\angle EAD = 30$. Here it is helpful to see that ΔDAE is isosceles (since the square and the equilateral triangle have a side in common). This means $m\angle AED = m\angle ADE = 75$. Finally, we compute $15 + 75 = 90$.

21. If you know the formula, you know *n* lines can produce $\frac{n(n+1)}{2} + 1$ regions in a plane. In this way, we produce the desires number of lines to be greater than or equal to 40. So we have $\frac{n(n+1)}{2} + 1 \ \geq 40 \Longrightarrow \frac{n(n+1)}{2} \geq 39 \Longrightarrow n(n+1) \geq 78$, and by inspection (knowing *n* must be an integer) we know $n = 9$ is the smallest adequate integer value. Another way is by creating a list, and noticing that the difference between the number of regions on two consecutive rows is increasing by 1 each time as you go down the list. (It won't take long to produce the answer.)

0 lines = 1 region
1 line = 2 regions
2 lines = 4 regions
3 lines = 7 regions …

22. When two distinct circles intersect, there will be a maximum of 2 intersection points. The maximum number of pairs of intersecting circles is $_{15}C_2 = \frac{15!}{2!(15-2)!} = \frac{15!}{2!13!} = \frac{15(14)}{2} = 105.$ Since we can have 2 intersection points for each of these pairs $105(2) = 210$.

23. The solid that gives us a maximum volume for a given surface area is a sphere. Then we have $SA_{sphere} = 4\pi r^2 = 36$. This produces the value $r = \frac{3}{\sqrt{\pi}}$. We then know the volume will be $V_{sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{\sqrt{\pi}}\right)^3 = \frac{4}{3}\pi\left(\frac{27}{\pi\sqrt{\pi}}\right) = \frac{36}{\sqrt{\pi}} = \frac{36}{n}$ so $n = \sqrt{\pi}$.

24. Using Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b, and c are side lengths and *s* is the semi-perimeter. Then $s = \frac{1}{2}(13+14+15) = \frac{1}{2}(42) = 21$ and $A = \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21(8)(7)(6)} = 84$.

25. Focus on the center of the coin, as trying to focus on more than that makes the problem unnecessarily complicated. The bottom of the box is 16 inches by 16 inches. Since the coin has radius 1, its center could never be closer than 1 inch from an edge. This means the center of the coin must lie somewhere inside of a 14 inch by 14 inch square. Then, since we are favoring an outcome where the coin is at least 2 inches from an edge, we know the center of the coin must be at least 3 inches from any edge. This produces a square with side length 10. So we have a favorable area of 100 inside a total considerable area of 196, or we can think $\left(\frac{10}{14}\right)^2 = \left(\frac{5}{7}\right)^2 = \frac{25}{49}$.