Precalculus Hustle Solutions
MAO National Convention 2015

Answers:

1. $x = \pm 3$
2. 35
3. $5/12$
4. $5\pi/2$
5. 24
6. 2
7. 29
8. $36/211$
9. $5/3$
10. 12
11. $\sqrt{2}/2$
12. $f(x) = \frac{2x^2-1}{3x}$
13. $2\sqrt{5}$
14. $3\sqrt{6} - 3\sqrt{2}$
15. $4/3$
16. 5
17. e
18. $6\sqrt{3}$
19. $\sqrt{13}$
20. 2
21. $20 \ln 403$
22. $\frac{24\sqrt{5}}{7}$
23. $(1 - 2\sqrt{3},\ \sqrt{3} + 2)$
24. $\sqrt{2}$
25. $1/8$

Solutions:

1. Putting everything on one side we get $|x|^2 - |x| - 6 = 0$. Factoring gives $|x| = 3,$ and $|x| = -2$. Clearly the latter doesn't work, so the answer is $\boldsymbol{x = \pm 3}$

2. Remove two willow trees to insert between the oak trees at the end to ensure they are not next to each other. This leaves seven trees, three of which are oak. We choose 3 spots for the oaks, giving us an answer of $\binom{7}{3} = \frac{7!}{4!3!} = \mathbf{35}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \right)$. Clearly this series is a telescoping sum. The only terms that do not cancel are $\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \mathbf{\frac{5}{12}}$

4. Using a double angle formula, we get $1 - 2\sin^2 x = \sin x \ \rightarrow\ 2\sin^2 x + \sin x - 1 = 0$. Factoring gives us $\sin x = \frac{1}{2}$ and $\sin x = -1$. Solving over the given domain, we get $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$. These sum to $\boldsymbol{5\pi/2}$

5. Multiply the inverse of $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ by both sides to isolate the unknown matrix. The inverse is $\frac{1}{-1}\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Therefore, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ -3 & -4 \end{bmatrix}$. $a + b - (c + d) = \mathbf{24}$

6. Using the conversion formulas to Cartesian coordinates, we get $x^2 + y^2 = 4x - 3$. This becomes $(x-2)^2 + y^2 = 1$. The difference between the maximum and minimum is the diameter which is $\mathbf{2}$

7. One could find both the cross product and the dot product to solve this question. Alternatively, notice that $(\boldsymbol{u} \cdot \boldsymbol{v})^2 + |\boldsymbol{u} \times \boldsymbol{v}|^2 = |\boldsymbol{u}|^2|\boldsymbol{v}|^2$. Therefore, the expression equals $(\boldsymbol{u} \cdot \boldsymbol{v})^2 + |\boldsymbol{u}|^2|\boldsymbol{v}|^2 = (-1)^2 + (2)(14) = \mathbf{29}$

8. For Jay to win in the first round, Robert and Ryan must roll a 4 or below and Jay must roll a 5 or 6. $P(\text{win round } 1) = \frac{2}{3} * \frac{2}{3} * \frac{1}{3}$. For Jay to win in the second round, everyone must have rolled a 4 or below the first round, and Robert and Ryan must the second round. $P(\text{win round } 2) = \left(\frac{2}{3}\right)^7 * \frac{1}{3}$. This becomes an infinite geometric series with ratio $\left(\frac{2}{3}\right)^5$. The sum is $\frac{\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)}{1-\left(\frac{2}{3}\right)^5} = \mathbf{\frac{36}{211}}$

9. The slopes of the lines are 2 and ½. Therefore, the smaller angle between the two lines is $\alpha = \arctan(2) - \arctan\left(\frac{1}{2}\right)$. $\tan(\alpha) = \frac{2-\frac{1}{2}}{1+1*\frac{1}{2}} = \frac{3}{4}$. We know $1 + \cot^2\alpha = \csc^2\alpha$. Plugging in, we get $\csc^2\alpha = 1 + \left(\frac{1}{\tan\alpha}\right)^2 = 1 + \left(\frac{4}{3}\right)^2 = \frac{25}{9}$. The angle is in the first quadrant, so $\csc\alpha = \mathbf{5/3}$

10. Use variants of the Pythagorean identity: $\sin^2 x + \cos^2 x = 1, \csc^2 x - \cot^2 x = 1, \tan^2 x - \sec^2 x = -1$. Applying these, we get $-4 + 25\sin^2 x$. Using the given information, we know $\sin x = 4/5$. So the answer is $-4 + 25 * \left(\frac{16}{25}\right) = \mathbf{12}$

11. $\tan(2\theta) = \frac{B}{A-C} = \frac{4}{0}$, so $2\theta = \frac{\pi}{2} \to \theta = \frac{\pi}{4}$. $\sin\left(\frac{\pi}{4}\right) = \mathbf{\frac{\sqrt{2}}{2}}$

12. Plugging in $1/x$, we get $2f\left(\frac{1}{x}\right) + f(x) = 1/x$. Subtract this from two times the original to get $3f(x) = 2x - \frac{1}{x}$. Therefore, $\boldsymbol{f(x)} = \mathbf{\frac{2x^2-1}{3x}}$

13. Notice that $\log_a b = \frac{1}{\log_b a}$. Set $\log_a b = x$ and we have

$$x^{16} + 2 + \frac{1}{x^{16}} = 49 \Rightarrow x^8 + 2 + \frac{1}{x^8} = 9 \Rightarrow x^4 + 2 + \frac{1}{x^4} = 5 \Rightarrow x^2 + \frac{1}{x^2} = \sqrt{5}.$$

$$x^6 + \frac{1}{x^6} = \left(x^4 + \frac{1}{x^4}\right)\left(x^2 + \frac{1}{x^2}\right) - x^2 + \frac{1}{x^2} = 3\sqrt{5} - \sqrt{5} = \mathbf{2\sqrt{5}}$$

14. Split the regular 24-gon up into 24 congruent triangles with 1 vertex at the center of the circle and 2 sides of length 1 (the radius). The included angle is 360/24 = 15°. The total area is $24\left(\frac{1}{2}*1*1*\sin(15°)\right) = 12*\frac{\sqrt{6}-\sqrt{2}}{4} = \mathbf{3\sqrt{6}-3\sqrt{2}}$. If using a half angle formula instead of difference, the answer $\mathbf{6\sqrt{2-\sqrt{3}}}$ is obtained.

15. Factor the bottom by sum of cubes to get $\frac{1}{\sin^2\frac{\pi}{12}-\sin\frac{\pi}{12}\cos\frac{\pi}{12}+\cos^2\frac{\pi}{12}}$. Use the Pythagorean identity and sine double angle formula to get $\frac{1}{1-.5\sin\frac{\pi}{6}} = \frac{1}{1-\frac{1}{4}} = \mathbf{4/3}$

16. $|3+4i| = \sqrt{3^2+4^2} = \mathbf{5}$

17. Notice that the interior sum is 2^n. Therefore, the sum is an infinite geometric series with first term ½ and ratio ½ . The sum is $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 = \ln A \Rightarrow A = \boldsymbol{e}$

18. The roots of this six degree equation form a regular hexagon. Clearly 2 is a root, so the circumradius of the hexagon is 2. Therefore, the area is $6*\left(\frac{1}{2}*2*2*\sin\left(\frac{360}{6}\right)\right) = \mathbf{6\sqrt{3}}$

19. Construct a triangle with these points and the origin and use the law of cosines with angle $\frac{\pi}{4}+\frac{\pi}{12}=\frac{\pi}{3}$. $d^2 = 3^2+4^2-2*3*4*\cos\left(\frac{\pi}{3}\right) \Rightarrow d = \mathbf{\sqrt{13}}$

20. Complete the squares under the radicals. Because constants are irrelevant as the limit approaches infinity, we get $x+\frac{3}{2}-\left(x-\frac{1}{2}\right) = \mathbf{2}$

21. Using the formula for continuously compounded interest, we plug in to get $2015 = 5e^{.05t} \Rightarrow \ln 403 = .05t \Rightarrow t = \mathbf{20\ln 403}$

22. The longest altitude goes to the shortest side (because Area is constant and $A = \frac{1}{2}bh$). Using Heron's to get Area, we have $\sqrt{12(3)(4)(5)} = \frac{1}{2}(7)h \Rightarrow h = \mathbf{\frac{24\sqrt{5}}{7}}$

23. Using a rotation matrix, we have $\begin{bmatrix}\cos 60° & -\sin 60° \\ \sin 60° & \cos 60°\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}x'\\y'\end{bmatrix}$ Solving, the new point is $(\mathbf{1-2\sqrt{3}},\ \mathbf{\sqrt{3}+2})$

24. We can use any point on one of the lines and use the point to line formula. Therefore, the shortest distance is the distance from the line $y = x + 1$ to (1, 0) which lies on $y = x - 1$ This is $\frac{1(1)-0(1)+1}{\sqrt{2}} = \sqrt{\mathbf{2}}$

25. $A = \frac{1}{2}ab\sin C$ and is maximized when $C = \frac{\pi}{2}$. Therefore, the maximum area is $\frac{1}{2}\cos 15° \sin 15° = \frac{1}{4}\sin 30° = \mathbf{1/8}$