Interschool Test--**SOLUTIONS** MAO Nationals 2015

1. \$30—Simplify this problem. Dividing by 30, we get a pig to be: $\frac{300}{30}=10$, and a goat to be $\frac{210}{30}=7$. It becomes evident that if you exchange 5 pigs for 4 goats, we get the smallest positive difference $5 \bullet 10 - 4 \bullet 7 = 1$. Since we originally divided by 30, we need to multiply y 30, and get the answer of 30.

2. 34— Let's set the middle (tens) digit first. The middle digit can be anything from 2-7 (If it was 1 we would have the hundreds digit to be 0, if it was more than 8, the ones digit cannot be even). If it was 2, there is 1 possibility for the hundreds digit, 3 for the ones digit. If it was 3, there are 2 possibilities for the hundreds digit, 3 for the ones digit. If it was 4, there are 3 possibilities for the hundreds digit, and 2 for the ones digit, and so on. So, the answer is $3(1+2)+2(3+4)+1(5+6)=34$.

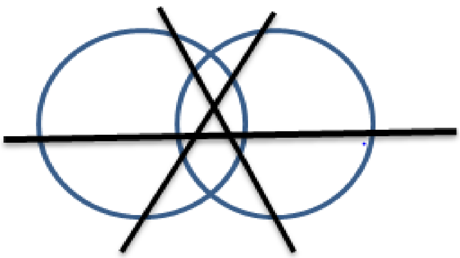
3. Saturday— There are 365 days in a year, plus 1 extra day if there is a Leap Day, in years that are multiples of 4 (with a few exceptions that don't affect this problem). Therefore, the number of days between Leap Day 2004 and Leap Day 2020 is: $16 \bullet 365 + 4 \bullet 1 = 5844$. Since the days of the week repeat every 7 days, $5844 \equiv -1 \pmod 7$, so 1 day before the 2004 Leap Day.

4. 95—The n*th* even natural nbr is given by 2n, so the 171^{st} even natural nbr is 2(171) = 342. The n*th* odd natural nbr is given by 2n – 1, so the 219^{th} odd natural nbr is 2(219) – 1 = 437. Their difference is 95.

5. 45 mph—Call the 2 digits in the mile markers that Calvin sees A and B. The 1^{st} mile marker is AB, the 2^{nd} mile marker is BA and the 3^{rd} mile marker is either A0B or B0A. Since the 1^{st} two mile markers are 2-digit numbers, this implies Calvin was driving less than 90 mph. So, the 3^{rd} mile marker has to be between 100 and 190. Logic dictates that the 1^{st} digit of the 3^{rd} mile marker must be 1, so the 1^{st} digit in the 1^{st} mile marker is also 1. In 2 hours, Calvin traveled from the 1^{st} mile marker to the 3^{rd} mile marker. His distance was therefore, (100 + B) – (10 + B) = 90. So, his average speed was 90/2 = 45 mph. The 3^{rd} mile marker must be of the form A0B, since 45 mph does check to mile markers 16, 61 and 106.

6. 167 days—All metric units are powers of 10, so a metric month has 365/10 = 36.5 days, a metric week has 365/100 = 3.65 days, a metric day has 365/1000 = 0.365 days. So, 4 metric months, 5 metric weeks and 8 metric days is 4(36.5) + 5(3.65) + 8(0.365) = 167.17 days. Rounded to the nearest day is 167.

7. 17 points—Begin with the 2 circles. Because they are not coincident, they have a maximum of 2 intersection points. When the 1st line is drawn, it can intersect each circle in 2 points. The 2nd line also can intersect each circle in 2 points and it can also intersect the 1st line. The 3rd line can intersect each circle in 2 points and can intersect each of the 1st 2 lines. The maximum number of intersection points, then, is 2 + 4 + 5 + 6 = 17. One such configuration is shown:



8. 371—A 3-digit number ABC in base 10 implies that ABC = 100(A) + 10(B) + 1(C). With the digits reversed in hexadecimal notation, the value is 256(C) + 16(B) +1(A). Setting these equations = gives 99A – 6B - 255C = 0. We know that A is a digit from 1 – 9. So, 99A will have a value less than 1000. Therefore C must have a value less than 4. Hence, the possible values of 255C are 255, 510, and 765. Also, B is a digit from 1 – 9, so 6B has a value less than 55. Now, 510 can be eliminated as a choice for 255C because 510 + 6B (regardless of B) will be less than 594, which is the value for 99A when A = 6. If we use similar reasoning with the 2 remaining values of 255C (255 & 765), it is found that A = 3 & B = 7, when C = 1. So, the number is 371 (base 10) = 173 (base 16).

9. 7652341—Because the largest 7-digit number is required, place 7 in the first position. Next place the greatest digit whose sum with 7 is a prime number, which is 6. The 3rd digit needs to be the largest digit whose sum with 6 is also a prime number, which is 5. The greatest digit remaining is now 4. Sadly, it cannot be the 4th digit because 5 + 4 = 9, which is not prime. Likewise, 3 cannot be the 4th digit, so 2 must be it. Similar reasoning for follows for the other digits and largest 7 digit number is 7652341.

10. 32 dollars

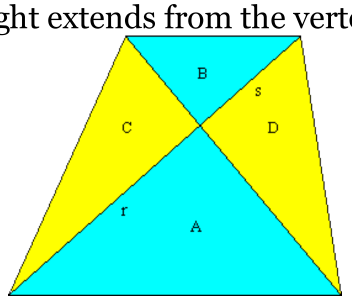
TRIVIA:

11. Aristocracy
12. 2 violins, 1 viola and 1 cello
13. Definite Article
14. Roentgen
15. Queue
16. Athens and Sparta
17. Classifies organisms
18. John Adams
19. 23
20. Zero

21. 7/13—Let p be the probability that student A wins. We consider the possible outcomes of the first two rolls. (Recall that each *roll* consists of the throw of two dice.) Consider the following mutually exclusive cases, which encompass all possibilities.
If the first roll is a 12 (probability 1/36), A wins immediately.
If the first roll is a 7 and the second roll is a 12 (probability 1/6 · 1/36 = 1/216), A wins immediately.
If the first and second rolls are both 7 (probability 1/6 · 1/6 = 1/36), A cannot win. (That is, B wins immediately.)
If the first roll is a 7 and the second roll is neither a 7 nor a 12 (probability 1/6 · 29/36 = 29/216), A wins with probability p.
If the first roll is neither a 7 nor a 12 (probability 29/36), A wins with probability p.
Note that in the last two cases we are effectively back at square one; hence the probability that A subsequently wins is p. Probability p is the weighted mean of all of the above possibilities. Hence p = 1/36 + 1/216 + (29/216)p + (29/36)p. Therefore p = 7/13.

22. $24 = \dfrac{6}{1 - \frac{3}{4}}$ The solution is unique.

23. $(\sqrt{A} + \sqrt{B})^2$—We will use the fact that the area of a triangle = ½ bh. Any side can serve as the base, and then the perpendicular height extends from the vertex opposite the base to meet the base



(or an extension of it) at right angles.
Consider the triangles with base r and s, and area C and B, respectively. These triangles have common height and collinear base; therefore r/s = C/B. Similarly, r/s = A/D. Hence AB = CD. Now consider the triangles with area A + C and A + D. These triangles have the same base and common height; hence A + C = A + D, and C = D. Hence C = D = $\sqrt{AB}$. So, the area of the trapezoid is

$$A + B + C + D = A + B + 2\sqrt{AB} = \left(\sqrt{A} + \sqrt{B}\right)^2 .$$

24. 273—Let the TD travelled downhill, on the level, and uphill, on the outbound journey, be x, y, and z, respectively. The time taken to travel a distance s at speed v is s/v. Hence, for the outbound journey x/72 + y/63 + z/56 = 4. While for the return journey, (we assume to be along the same roads) x/56 + y/63 + z/72 = 14/3. We are not asked for the values of x, y, and z, individually; but for the value of x + y + z. Multiplying both equations by the LCM of denominators 56, 63, and 72, we obtain 7x + 8y + 9z = 4 · 7 · 8 · 9 and 9x + 8y + 7z = (14/3) · 7 · 8 · 9. Now it is clear that we should add the equations, yielding 16(x + y + z) = (26/3) · 7 · 8 · 9. Therefore x + y + z = 273; the distance between the two towns is 273 miles.

25. 7 miles—Let a point P denote the car, A denote Alpha, B denote Beta, and C denote Gamma. Clearly P must be on the opposite side of AB to C, for otherwise we could reflect P in AB, thereby increasing CP, while keeping AP and BP the same. Also, P must be on the same side of AC as B, for otherwise we could reflect P in AC, and then extend AB so that BP = 4, thereby increasing CP. Similarly, P must be on the same side of AB as C. Hence quadrilateral APBC is convex, and with diagonals AB and CP, so that we apply Ptolemy's Inequality, which states that: $AB \cdot CP \leq AP \cdot BC + BP \cdot AC$, with equality if, and only if, APBC is cyclic. Since AB = BC = AC, we get $CP \leq AP + BP = 7$, the maximum possible distance of the car from Gamma is 7 miles.

26. 11—First,draw a line from C to the center of the circle O. Area $\triangle ABC$ $\sqrt{[22 \cdot (22-11) \cdot (22-13) \cdot (22-20)]} = 66$. Let the radius of the semicircle be r, so that OD = OE = r. Then area of $\triangle AOC = \frac{1}{2} \cdot AC \cdot r = 11r/2$. Similarly, area of $\triangle BOC = \frac{1}{2} \cdot BC \cdot r = 13r/2$. Since area of $\triangle ABC$ = area of $\triangle AOC$ + area of $\triangle BOC$, we have 66 = 12r, and hence r = 11/2. Therefore, the diameter of the semicircle is 11 units.

27. 1.83×10^{-37} --This problem is a counting exercise. Consider the four types of cubes upon disassembly: a. 8 cubes with three faces painted black; b. 12 cubes with two black faces; c. 6 cubes with one black face; d. 1 completely white cube. Each cube of type (a) must be oriented in one of three ways, giving 3^8 possible orientations. Next, each corner cube must be placed in one of eight corners, giving 8! possible arrangements. Thus we have $3^8 \cdot 8!$ possibilities in all. Similarly, each cube of type (b) must be oriented in one of two ways, giving 2^{12} possible orientations. Then, each edge cube must go to one of 12 edges, giving 12! poss. arrangements. Thus we have $2^{12} \cdot 12!$ possibilities in all. Also, each cube of type (c) has four possible orientations, and may be placed in one of six positions, yielding $4^6 \cdot 6!$ possibilities. Finally, the 1 white cube of type (d) may be oriented in 24 ways (four ways for each face). Thus the total number of correct reassemblings is $a = 3^8 \cdot 8! \cdot 2^{12} \cdot 12! \cdot 4^6 \cdot 6! \cdot 24$. To find the total number of *possible* reassemblings, consider that each cube may be oriented in 24 ways, and there are 27! possible arrangements of the cubes, giving $b = 24^{27} \cdot 27!$ possibilities in all. Therefore, the probability that the outside of the reassembled cube is completely black is $a/b = 1/(2^{56} \cdot 3^{22} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23) = 1/5465062811999459151238583897240371200 \approx 1.83 \times 10^{-37}$.

28. 5000—If the path is straightened out into a line, its area is that of the square, 100x100 square paces. Dividing by the path width by 2 gives the solution.

Interschool Test—**SOLUTIONS**-p5 MAO Nationals 2015

29. No other chair is required as there are only seven people. A Grandfather married to Grandmother. They have a son who is married to a woman. Together, they have three children - One son and two daughters. These are the only seven people.

30. None— There is no number other than 7 that is a prime followed by a cube.

31. ½--Option 1: If any of the first 99 people sit in Nathan's seat, Nathan will not get to sit in his own seat. Option 2: first 99 people If any of the sit in Catherine's seat, Nathan will get to sit in his seat.

32. Two—Because the quotient has 5 digits but only 3 subtractions were performed, 2 digits of the quotient must be zero. These digits cannot be the 1st or last. So, K = L = 0. Note that J x AB is NPQ, but 8 times AB is TU, so J must be 9. M times AB is YOZ (a 3-digit number), so M is also 9. We see that AB times 8 has a 2-digit product, and AB times 9 has a 3-digit product. So, AB = 12. Since CDEFGHI equals AB times JK8LM plus 1, the dividend must be 1089709.

33. 18—Let $u=2^x$, So, $u^2 + u^{-2} = 7$ and $u^2 + 2 + u^{-2} = 9 = (u + u^{-1})^2$ & $u + u^{-1} = 3$. So, $u^3 + u^{-3} = (u + u^{-1})(u^2 - 1 + u^{-2}) = (3)(7-1) = 18$

34. Begin by multiplying through by cos(3x): $\cos(3x)\sin(2x)+\sin(3x)+\cos(3x)\cos(4x)-2\cos(3x)=0$. Remember, $\cos(3x)=\cos(2x+x)=\cos(2x)\cos(x)-\sin(2x)\sin(x)= \cos(x)-2\cos(x)\sin^2(x)-2\sin^2(x)\cos(x)$ and $\sin(3x)=\sin(2x+x)=\sin(2x)\cos(x)+\cos(2x)\sin(x)= 2\cos^2(x)\sin(x)+\sin(x)-2\sin^3(x)$and $\cos(4x)=\cos(2(2x))=1-2\sin^2(2x)= 1-8\sin^2(x)\cos^2(x)$. Hence, x= 15 or 75.

35.Zero—From the initial information, we can determine that -B = -A, so A and B are equal. Furthermore, we can determine that -C and B - D are equal. Thus, the determinant is equivalent to the top row of A and A, and the bottom row of C and C, so the determinant is zero.

36. 1—The pattern of the units digit is 2, 4, 8, 6, 2, 4, 8, 6,…. If 11,213 is divided by 4, the remainder is 1, so the units digit of 2^{11213} is the same as the units digit of 2^1, which is 2. So, the units digit of $2^{11213} - 1$ is 2 – 1 = 1.

37. 7—There is 1 denomination with no corners removed and there is 1 denomination will all 4 corners removed. There is just 1 distinguishable bill with 3 corners removed, and there is likewise 1 bill with 1 corner removed. Removing 2 corners, gives several possibilities: adjacent corners (both on the width) can be removed; adjacent corners (both on the length) can be removed; and opposite corners can be removed. In total, 7 possible denominations.

38. $72\sqrt{2}$ -The solid of intersection is an octahedron. Because the pyramids are identical in shape, their lateral faces intersect so that all edges of the octahedron have length 6 cm. Finding the volume of the octahedron entails finding the volume of one of the smaller pyramids that form half of it and doubling it. We find the slant height by the Pythagorean Thm, which is $\sqrt{6^2 - 3^2} = 3\sqrt{3}$. The height is also found by the Pythag. Thm., which is $\sqrt{\left(3\sqrt{3}\right)^2 - 3^2} = 3\sqrt{2}$. The area of the base is 6 x 6 = 36, so V = $\frac{1}{3}(36)(3\sqrt{2}) = 36\sqrt{2}$. Doubling this gives the volume of the octahedron.

39. 99 degrees—The greatest common factor of 90, 100 and 1600 is 10. So, the 3 measures will all have integer values when degrees are a multiple of 90/10 = 9, when grads are a multiple of 100/10 = 10, and when mils are a multiple of 1600/10 = 160. Since the next number of degrees greater than 90 that all 3 have integer units, just add 9 to 90 and get 99.

40. e^i —Writing -1 in polar form gives $e^{i\pi}$. $(-1)^{-1} = -1$, so $\sqrt[\pi]{-1} = \left(e^{i\pi}\right)^{\frac{1}{\pi}} = e^i$