Congratulations! You've just been hired as a mathematical consultant by Mu Alpha Theta Incorporated (MAΘ Inc.). Before you begin getting case assignments, you must first complete the MAΘ Inc. certification exam below. Remember NOTA means none of the above answers is correct and problems will specify the units of answer choices if necessary. Good luck!

MAΘ Inc. Certification Exam:

1. Ankit's position with respect to time is described by the function $x(t) = t\sin t$. Which of the following describes Ankit at $t = \pi/2$?

   A. Speeding up B. Slowing down C. Stopped D. Neither speeding up nor slowing down E. NOTA

2. What is the side length of an expanding equilateral triangle when the rate of change of its perimeter with respect to time is equal to the rate of change of its área with respect to time?

   A. $2\sqrt{3}$ B. $\frac{2\sqrt{3}}{3}$ C. $\frac{2\sqrt{3}}{9}$ D. $\frac{\sqrt{3}}{2}$ E. NOTA

3. Use Newton's Method to approximate a zero of $f(x) = x^3 + x$ starting at $x_0 = 1$ and taking two steps. (Find x_2).

   A. $1/2$ B. $1/3$ C. $1/7$ D. $-7/9$ E. NOTA

4. A ten foot ladder that stands vertical against a wall begins to fall so that its top slides down the wall at 1 ft/s and its base moves horizontally along the ground. After two seconds, how fast is the base moving away from the wall? Answers are in ft/s.

   A. $4/3$ B. $\sqrt{19}/19$ C. 3 D. $\sqrt{19}$ E. NOTA

5. A concert venue knows that if it sells tickets at price x (in dollars), then $10{,}000 - 10x$ people will come. The singer charges the venue $\$1{,}000$ plus $\$30$ per person to perform. How much should the venue charge per ticket to maximize profit?

   A. $\$498.50$ B. $\$500$ C. $\$515$ D. $\$530$ E. NOTA

6. In the 100 prisonsers problema, the prisoners' probability of survival is $1 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$. Find the probability of survival as n approaches infinity.

   A. 1 B. $\ln 2$ C. $1 - \ln 2$ D. $1 - \frac{1}{2}\ln 2$ E. NOTA

7. Find the values of x for which the series below converges.

$$\sum_{n=1}^{\infty} (\ln(2x))^n$$

A. $\left(\frac{2}{e}, 2e\right)$ B. $\left[\frac{1}{2e}, \frac{e}{2}\right)$ C. $\left[\frac{2}{e},\ 2e\right]$ D. $\left(\frac{1}{2e}, \frac{e}{2}\right)$ E. NOTA

8. Evaluate the sum from the previous problem (#7) when $x = \sqrt{e}/2$.

A. $1/2$ B. 1 C. 2 D. 4 E. NOTA

9. A particle moves in the xy plane with position equations $x(t) = e^{-t}$, $y(t) = t^2$. Evaluate $\frac{d^2y}{dx^2}$ at $t = 2$.

A. $-2e^2$ B. $2e^4$ C. $-6e^2$ D. $6e^4$ E. NOTA

You are now a full-fledged, certified MAΘ Inc. mathematical consultant. Well done! Your first case assignment…

Client 1: Austin Powers

International Man of Mystery, Austin Powers, has requested your help to take down Dr. Evil.

10. Dr. Evil, a vehement hater of the cold, is building a matter vaporizer to destroy the cold regions of the Earth. What fraction of the Earth's volume lies below the 60th parallel in the southern hemisphere or above the 60th parallel in the northern hemisphere? (The 60th parallel runs parallel to the equator and 60 degrees north or south of the equator along the arc of a great circle passing through both poles. Assume the Earth is a sphere.)

A. $5/32$ B. $5/16$ C. $\frac{16-9\sqrt{3}}{32}$ D. $\frac{16-9\sqrt{3}}{16}$ E. NOTA

11. Dr. Evil's dimensionless escape car departs from rest at point P, going in a straight line, with a constant acceleration of $1.5\frac{m}{s^2}$. Austin arrives at point P six seconds after Dr. Evil leaves P, chasing after Dr. Evil. What is the minimum constant speed (in m/s) at which Austin (also a point) can run and still catch Dr. Evil before he gets away forever?

A. 9 B. 18 C. 27 D. 36 E. NOTA

12. Unfortunately, despite your calculations, Dr. Evil escapes, so Austin fires a projectile. What is the sine of the maximum angle above the horizontal at which he can fire if the distance between Austin (not moving) and the projectile is never decreasing? Use $-10\frac{m}{s^2}$ as the acceleration due to gravity. Air resistance is negligible, and don't worry about actually hitting the car for the purposes of this problem.

A. $2\sqrt{2}/3$ B. $\sqrt{2}/3$ C. $2/9$ D. $2/3$ E. NOTA

13. Austin decides to shoot the projectile at an angle of 1 radian above the horizontal at $120\,\frac{m}{s}$. Approximate its initial vertical velocity (in m/s) with a fifth degree Maclaurin Polynomial.

A. 44 B. 65 C. 101 D. 141 E. NOTA

14. It's a hit! The car begins to leak coolant at a constant rate $r > 0$. If there were initially C units of coolant in the tank and the car's velocity is directly proportional to the amount of coolant left, how far will the car travel from when the coolant began leaking until it comes to rest? Let k be the constant of proportionality.

A. $\frac{kC^2}{r}$ B. $\frac{kC^2}{2r}$ C. $\frac{kC^2}{r^2}$ D. $\frac{kC^2}{2r^2}$ E. NOTA

15. To celebrate his victory over Dr. Evil, Austin wants a groovy new logo in the shape of a flower with the equation $r(\theta) = a\sin(n\theta)$. How many petals could Austin's logo NOT have?

A. 7 B. 12 C. 17 D. 18 E. NOTA

Client 2: Mu Alpha Ikea

The hip, new household store, Mu Alpha Ikea, wants your help in unveiling its latest mathematically inspired beverage coaster line.

16. One of Mu Alpha Ikea new coaster designs is the shape bounded by one arch of a cycloid and the x axis. What is the perimeter of this shape? The cycloid is described parametrically by $x(t) = 2(t - \sin t)$, $y(t) = 2(1 - \cos t)$.

A. $16 + 4\pi$ B. $16 + 2\pi$ C. $\frac{\pi^2}{2} + 2\pi$ D. $\pi^2 + 2\pi$ E. NOTA

17. Analysts at Mu Alpha Ikea have determined that the profit function for this coaster is $P(x) = 3x^{2/3}$, where x is the number of units. Approximate $P(100)$ using differentials, starting at $P(125)$.

A. 60 B. 65 C. $71\frac{1}{3}$ D. 73 E. NOTA

18. Another one of their new designs is the shape bounded by the graphs $y = x^2$ and $y = 4$. What is the distance from the origin to the centroid of this region?

A. $12/5$ B. $12\sqrt{2}/5$ C. 2 D. $24/5$ E. NOTA

19. The cost function for this design is $C(x) = 2\ln(2x)$. The marginal cost at x is $C'(x)$. What is the marginal cost at $x = 50$?

A. $1/25$ B. $1/50$ C. $1/100$ D. $1/200$ E. NOTA

20. A third design involves a trapezoid inscribed in a semicircle with radius 1 such that one base of the trapezoid lies on the diameter of the semicircle. What is the maximum area this trapezoid could have?

A. 1 B. 2 C. $3\sqrt{3}/4$ D. $\sqrt{2}$ E. NOTA

21. For this design, the demand function $D(t), t \in (0, 12]$ is $D(t) = \frac{\ln t}{t}$. Find the average value of demand during a given year, where t is in months.

A. 6 B. $\frac{(\ln 12)^2}{2}$ C. $\frac{(\ln 12)^2}{12}$ D. $\frac{(\ln 12)^2}{24}$ E. NOTA

22. Finally, as part of their deluxe package, Mu Alpha Ikea has a 3d coaster that is in the shape of the ellipse $\frac{x^2}{4} + y^2 = 1$ rotated about the line $x = 3$. Find the volume of this solid.

A. 3π B. 6π C. 12π D. 24π E. NOTA

23. Occasionally coasters malfunction leading to catastrophic beverage failure (CBF). If X is a random variable representing the number of days until CBF and has the probability distribution function below, find $P(X > 10)$. (Hint: find α first by integrating over all possible values of x. What should this probability equal?).

$$f(x) = \begin{cases} \alpha e^{-x/200} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

A. $e^{-1/20}$ B. $1 - e^{-1/20}$ C. $1/20$ D. $1/200$ E. NOTA

Client 3: Busch Gardens

Busch Gardens Tampa Bay has hired you to help introduce a new Lion habitat to the park. They ask you to help them prepare for the exhibit and help them overcome any issues that arise.

24. Before the new habitat is completed, you notice an invasive species of plant. If the seed dispersal density for the plant is given by $f(x) = 2e^{-2x}$ where x is the distance from the plant, what is the expected mean distance at which the seeds are dispersed?

A. $1/2$ B. 1 C. 2 D. 8 E. NOTA

Use the following information for questions 25-27:
Busch Gardens needs to figure out how fast the lion population will grow in order to build an adequate habitat. They initially introduce 10 new lions into the habitat. Their population $P(t)$ grows at a rate directly proportional to $P(t)\big(600 - 15P(t)\big)$, and there are 15 lions after one year ($t = 1$).

25. Find $\lim_{t\to\infty} P(t)$.

A. 20 B. 40 C. 60 D. 600 E. NOTA

26. When is the population growing the fastest?

A. $P = 0$ B. $P = 10$ C. $P = 20$ D. $P = 40$ E. NOTA

27. Which of the following values is closest to $P(2)$?

A. $P = 19$ B. $P = 20$ C. $P = 21$ D. $P = 40$ E. NOTA

Use the following information for questions 29-30
A sick lion requires an antibiotic that decays in the body at a rate directly proportional to the amount present. A veterinarian knows that when 60mg is administered, there is 20mg left after 1 hour.

28. What is this antibiotic's half life (in hours)?

A. $\log_{3/2} 2$ B. $\log_{3/2} 4$ C. $\log_3 2$ D. $\log_3 4$ E. NOTA

29. The veterinarian wants to keep the drug at 100mg in an infected animal. If the initial dose is 100mg, at what constant rate (in mg/hour) should an IV administer the antibiotic to maintain this amount in the body?

A. 50 B. 100 C. $50 \ln 3$ D. $100 \ln 3$ E. NOTA

30. Unfortunately, one day a Busch Gardens veterinarian finds that one of their lions passed away during the night. The veterinarian finds that the lion's temperature is $81^O F$ when he arrives in the park at 6 AM and $66^O F$ four hours later. If the habitat is a constant $61^O F$ and the lion's normal body temperature is $101^O F$, what time did the lion die, to the nearest hour? Assume the body observed Newton's law of cooling beginning at time of death.

A. 12 AM B. 2 AM C. 4AM D. 5AM E. NOTA