Answers:

1. **B**	7. **D**	13. **C**	19. **A**	25. **B**
2. **A**	8. **B**	14. **B**	20. **C**	26. **C**
3. **C**	9. **D**	15. **D**	21. **E**	27. **C**
4. **A**	10. **D**	16. **A**	22. **E**	28. **C**
5. **C**	11. **B**	17. **B**	23. **A**	29. **D**
6. **C**	12. **A**	18. **A**	24. **A**	30. **C**

Solutions:

1. Taking the first and second derivatives we get the functions $v(t) = \sin t + t\cos t$ and $a(t) = 2\cos t - t\sin t$. $v\left(\frac{\pi}{2}\right) = 1$, $a\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$. This means Ankit is slowing down. **B**

2. We know $A = \frac{s^2\sqrt{3}}{4}$ and $P = 3s$. Deriving, we get $\frac{dA}{dt} = \frac{s\sqrt{3}}{2}\frac{ds}{dt}$ and $\frac{dP}{dt} = 3\frac{ds}{dt}$. We are given that the area and perimeter are changing at the same rate, so we can substitute $3\frac{ds}{dt}$ for $\frac{dA}{dt}$. $3\frac{ds}{dt} = \frac{s\sqrt{3}}{2}\frac{ds}{dt} \rightarrow s = 2\sqrt{3}$. **A**

3. In Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For this function, we have $x_{n+1} = x_n - \frac{x_n^3 + x_n}{3x_n^2+1}$. $x_1 = 1 - \frac{2}{4} = 1/2$. $x_2 = \frac{1}{2} - \frac{5/8}{7/4} = 1/7$. **C**

4. The ladder can be viewed as the hypotenuse of a right triangle while sliding down the building, so we have $x^2 + y^2 = 100$. After two seconds, the top will have moved 2ft down the building, giving us a leg of $y = 8\text{ft} \rightarrow x = 6\text{ft}$. We derive the Pythagorean theorem to get $x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \rightarrow 6\frac{dx}{dt} + 8(-1) = 0 \rightarrow \frac{dx}{dt} = 4/3$. **A**

5. Profit is given by revenue – cost. $P = x(10{,}000 - 10x) - \left(1{,}000 + 30(10{,}000 - 10x)\right)$. Simplify to get the parabola $P = -10x^2 + 10{,}300x - 301{,}000$. The maximum is at $x = -\frac{b}{2a} = \$515$ **C**

6. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}} = \int_0^1 \frac{1}{x+1}\,dx = \ln 2$. Thus, the answer is $1 - \ln 2$. **C**

7. One could use the ratio test or recognize this as an infinite geometric series. Both yield $|\ln(2x)| < 1 \rightarrow -1 < \ln(2x) < 1 \rightarrow \frac{1}{e} < 2x < e \rightarrow \frac{1}{2e} < x < \frac{e}{2}$. **D**

8. Notice that this value is in the desired range. We use the formula for an infinite geometric series: $\sum_{n=1}^{\infty}\left(\ln\left(\frac{2\sqrt{e}}{2}\right)\right)^n = \frac{\ln\sqrt{e}}{1-\ln\sqrt{e}} = 1$ **B**

9. For these parametric equations, we have $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 2t(-e^t)$, and $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(\frac{d}{dt}(-2te^t)\right)(-e^t) = 2e^{2t}(t+1)$. At $t = 2$, this is $6e^4$. **D**

10. For the purposes of this problem, let's treat the Earth as a sphere of radius 2. This means that the denominator of our fraction is $\frac{4}{3}\pi r^3 = \frac{32\pi}{3}$. Using trig, we see that the 60th parallel cuts this sphere $\sqrt{3}$ above/below from the center. We can find the volume of one of these regions by rotating a portion of the first quadrant quarter circle $y = \sqrt{4-x^2}$ about the x axis using the disk method: $V = 2\pi\int_{\sqrt{3}}^{2}\left(\sqrt{4-x^2}\right)^2 dx = 2\pi\left(4x - \frac{x^3}{3}\Big|_{\sqrt{3}}^{2}\right) = 2\pi * \frac{16-9\sqrt{3}}{3}$. Thus, our fraction is $\frac{2\pi*\frac{16-9\sqrt{3}}{3}}{\frac{32\pi}{3}} = \frac{16-9\sqrt{3}}{16}$. **D**

11. We are given Dr. Evil starts at rest, so $v_D(0) = 0$. We can treat P as the reference, so $x_D(0) = 0$. Thus, we integrate Dr. Evil's acceleration twice to get $x_D(t) = \frac{3}{4}t^2$. Austin runs at a constant speed but starts six seconds later, so $v_A(t) = v(t-6)$ where v is the speed at which Austin runs. We are trying to find the minimum v for which they meet, or in other words, for which $\frac{3}{4}t^2 = v(t-6) \to 3t^2 - 4vt + 24v = 0$ has a solution. The slowest Austin can run is when he only catches Dr. Evil once (versus passing then being passed). This equation has one solution when its discriminant is zero: $0 = b^2 - 4ac = 16v^2 - 4(24)(3)v = 0 \to v^2 - 18v = 0 \to v = 18, v = 0$. Only the positive answer makes sense in the context of this question. **B**

12. If a projectile is fired at velocity v and angle θ above the horizontal, the initial y component of the velocity is $v\sin\theta$ and the initial x component is $v\cos\theta$. There is no acceleration in the x direction, so $x(t) = vt\cos\theta$. Integrating the constant acceleration of gravity, we get $y(t) = vt\sin\theta - 5t^2$. The distance between Austin and the projectile is $D(t) = \sqrt{\left(x(t)\right)^2 + \left(y(t)\right)^2} = \sqrt{v^2t^2 - 10vt^3\sin\theta + 25t^4}$ using the Pythagorean identity. This distance will never decrease if its derivative is never negative. We can ignore the square root for this part since it is always nonnegative. $D'(t) = t(100t^2 - (30v\sin\theta)t^2 + 2v^2)$. Since t is never negative, this function is nonnegative if the parabola $100t^2 - (30v\sin\theta)t^2 + 2v^2$ never crosses the x axis. This means its discriminant is less than or equal to zero: $900v^2\sin^2\theta - 4(100)(2v^2) \le 0 \to \sin^2\theta \le \frac{8}{9} \to \sin\theta \le \frac{2\sqrt{2}}{3}$. **A**

13. Initial vertical velocity is $120\sin 1 \approx 120\left(1 - \frac{1}{3!} + \frac{1}{5!}\right) = 120 - 20 + 1 = 101$ **C**

14. The amount of coolant left at time t is $C(t) = C - rt$. We are given $v(t) = k(C - rt)$. This tells us that $v = 0$ when $t = C/r$. Integrate velocity to find distance traveled ($x(0) = 0$ since we are only concerned distance while coolant is leaking): $x(t) = kCt - \frac{1}{2}krt^2$. Plugging in $t = C/r$, we get $\frac{kC^2}{r} - \frac{1}{2} * \frac{kC^2}{r} = \frac{kC^2}{2r}$. **B**

15. If n is odd, the flower has n petals, and if n is even, the flower has $2n$ petals. This means, a flower can have any odd number of petals, but only an even number of petals that is a multiple of 4. Thus, 18 is an invalid number of petals. **D**

16. The cycloid has one arch between 0 and 2π, so the total perimeter is the curve length between two points plus the segment along the x axis with length 2π. The curve length is $\int_0^{2\pi} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} dt = 2\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = -8\cos\frac{\theta}{2}\Big|_0^{2\pi} = 16$. The total perimeter is $16 + 4\pi$. **A**

17. $P(100) \approx P(125) - 25\left(P'(125)\right) = 75 - 25\left(\frac{2}{5}\right) = 65$. **B**

18. The axis of symmetry is at $x = 0$, so $\bar{x} = 0$. $\bar{y} = \frac{1}{2A}\int_{-2}^{2}(16 - x^4)dx = \frac{3}{64}\left(16x - \frac{x^5}{5}\Big|_{-2}^{2}\right) = \frac{3}{64}\left(\frac{4(64)}{5}\right) = \frac{12}{5}$. **A**

19. $C'(50) = \frac{2}{50} = \frac{1}{25}$. **A**

20. To achieve maximum area, one base must be along the diameter with a vertex at each side. This base will have length 2, and other base will have length 2x. The height of the trapezoid, given by the y coordinate of the semicircle is $\sqrt{1 - x^2}$. Thus, we are maximizing the function $A = \frac{1}{2}(2 + 2x)\sqrt{1 - x^2} = \sqrt{1 - x^2} + x\sqrt{1 - x^2}$ on $x \in [0, 1]$. Take the derivative and set equal to zero and solve: $A' = -\frac{x}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1 - x^2} = 0 \to x = \frac{1}{2}$. Plug back into the area function to find the maximum area of $\frac{3\sqrt{3}}{4}$. **C**

21. Average value = $\frac{1}{12-0}\int_0^{12}\frac{\ln t}{t}dt$. Notice that $\frac{d}{dt}(\ln t) = \frac{1}{t}$. Integrate to get $\frac{1}{12}\left(\frac{(\ln(t))^2}{2}\Big|_0^{12}\right) = \lim_{a\to 0}\frac{1}{12}\left(\frac{(\ln(t))^2}{2}\Big|_a^{12}\right) = \frac{1}{24}\lim_{a\to 0}((\ln 12^2)^2 - (\ln a)^2) = \frac{1}{12}((\ln 12^2)^2 - \infty)$. Diverges. **E**

22. Pappus' theorem tells us that the volume of a rotated 3D figure is the distance its centroid travels times the cross sectional area. ($V = 2\pi dA$) The area of this ellipse is $\pi ab = 2\pi$, and it's centroid is at $(0, 0)$ due to symmetry. Thus, $V = \left(2\pi(3 - 0)\right)(2\pi) = 12\pi^2$. **E**

23. We can find α by realizing that the sum of all probabilities for all valid distances is equal to 1: $1 = \int_0^{\infty}\alpha e^{-x/200}dx = \alpha(0 + 200) \to \alpha = 1/\,200$. Now we can find the probability that

$X > 10$ by computing $1 - P(0 < X < 10) = 1 - \frac{1}{200}\int_0^{10} e^{-\frac{x}{200}}dx = 1 - \left(-e^{-\frac{1}{20}} + 1\right) = e^{-\frac{1}{20}}$. **A**

24. $EV = \int_0^{\infty} xf(x)dx = 2\int_0^{\infty} xe^{-2x}dx$. Using integration by parts or tabular method, this integral evaluates to $-xe^{-2x} - \frac{e^{-2x}}{2}\Big|_0^{\infty} = 1/2$. **A**

25. The derivative $\frac{dP}{dt} = kP(t)\big(600 - 15P(t)\big)$ equals zero at $P = 40$. This means that the population will stop growing at 40 lions. **B**

26. Maximize the derivative $P' = kP(600 - 15P)$ which is a parabola. This means its maximum is at $P = -\frac{b}{2a} = 20$. Alternatively, you can take the derivative and set it equal to zero. Notice this is half of the carrying capacity. **C**

27. The solution to the differential equation is of the form $P(t) = \frac{40}{1+Ce^{kt}}$. Using $P(0) = 10$ and $P(1) = 15$, we find that $C = 3$ and $k = \ln\left(\frac{5}{9}\right)$. We now plug in $t = 2$ and get $\frac{270}{13} \approx 20.7$ **C**

28. $\frac{dA}{dt} = kA \rightarrow A(t) = Ce^{kt}$. Plugging in $A(0)$ and $A(1)$ we get $C = 60$, $k = -\ln 3$. Now we solve $30 = 60e^{-\ln 3(t)} \rightarrow \log_3 2 = t$ **C**

29. We know from the previous question that $\frac{dA}{dt} = -\ln 3\, A + r$. We want the drug to stay at a constant level after the initial dose: $\frac{dA}{dt} = 0 \rightarrow -\ln 3\ (100) + r = 0 \rightarrow r = 100\ln 3$. **D**

30. Newton's law of cooling gives us that $\frac{dT}{dt} = k(T - 61) \rightarrow T = Ce^{kt} + 61$. We are given that $T(0) = 101$ which means $C = 40$. We also know $T(t_1) = 81$, and $T(t_1 + 4) = 66$, giving us the two equations:

$$20 = 40e^{kt_1} \text{ and } 5 = 40e^{k(t_1+4)} \rightarrow$$

$$-\ln 2 = kt_1 \text{ and } -3\ln 2 = kt_1 + 4k.$$

Solving gives us $= -\frac{\ln 2}{2}$, $t_1 = 2$. If 6AM is $t = 2$, then $t = 0$ is at 4AM. **C**