Mu Areas and Volumes Nationals 2015 Solutions

1.These are cone of radius and height 2 and a hemisphere of radius 2. Therefore the volume is 8π. Answer D.

2. The area of the sector is $\frac{\theta}{2}$ (because the radius is 1) and the area of the triangles is ½ $tan\theta$ and ½ $sin\theta$. Answer A.

3. Writing the inequality ½ $sin\theta < \frac{\theta}{2} <$ ½ $tan\theta$ and then multiplying by 2, reciprocating all the terms and multiplying by $sin\theta$ gives you $cos\theta < \frac{sin\theta}{\theta} < 1$. Answer B.

4. $h = x^2, l = 5x^2, w = dx. \int_0^3 5x^4 dx = 243$ Answer D.

5. One thing this represents is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx = 2$. Answer C.

6. Since no value of the radius is given and the rate is varying, the answer cannot be determined. Answer D.

7. Making a diagram with the line passing through the point and labeling the x-intercept (x, 0) and the y-intercept (0, y) allows you to create the equation $\frac{-y}{x} = \frac{-4}{x-2}$ which simplifies to $y = \frac{4x}{x-2}$. The area of the triangular region R is given by $A = \frac{1}{2}xy = \frac{2x^2}{x-2}$ therefore $A' = \frac{2x^2 - 8x}{(x-2)^2}$ which equals zero at x = 0 and 4 the latter of which represents the minimum. Plugged back into the formula for A we get 16. Answer A.

8. Answer C.

9. Answer B

10. This rose curve traces itself one full time on the interval [0, π]. $\int_0^{\pi} \frac{9}{2} sin^2(5\theta) d\theta = \frac{9\pi}{4}$ Answer B.

11. This is an ellipse, the area for which is given by $A = \pi ab = 2\pi^2$ Answer D.

12. Using the shell method you get $\int_0^1 2\pi(e^x - 1)(1 - x) = \pi(2e - 5)$ Answer A.

13. $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ So when r = .01 $\frac{dr}{dt} > \frac{dA}{dt}$ but as r increases it becomes smaller. Answer B.

14. All of these use rectangles with heights taken from some point within the rectangle or on it's edge, therefore all are valid. Answer E.

15. p represents the perimeter and q represents the area. As n gets large these approach the circumference and area of the circle respectively. Therefore $\frac{p}{q} = 2R^{-1}$ Answer C.

16. The area is found by taking the absolute value of the determinant of a 3x3 matrix containing the components of the three vectors formed by the edges, which in this case is 34. Answer D.

17. This can be computed but one can also realize this is the function x^3 translated 1 unit to the left and since the limits are also translated this is a case of an integral of an odd function from –a to a, which is 0. Answer E.

18. $A = \frac{1}{2}absin\theta$ which achieves a max at 90 degrees. Therefore A = 45. Answer A.

19. Whenever there is a maximum in the function, the rate of change is 0. Answer E.

20. This area can be expressed as a geometric series of ratio $\frac{1}{9}$ and a first term of 4. Therefore the area is $\frac{9}{2}$. Answer A.

21. Number III should be obvious, but since it's a right triangle, numbers I and II also work. Answer D.

22. $l = \sin(x), h = \cos(x), w = dx$ Therefore $V = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = \frac{1}{2}$ Answer B.

23. The circle that circumscribes this triangle is centered at the origin and has radius of 5 (if this is not obvious it can be found be constructing two perpendicular bisectors and getting their intersection point). Therefore the area is 25π. Answer D.

24. Answer B.

25. The midsegment of a triangle is half the length of the parallel side. Therefore each side of ABCD is half that of ROFL and the ratio would be 4:1. Answer C.

26. The polynomial is factorable into $\left(\theta - \frac{\pi}{2}\right)\left(\theta - \frac{2\pi}{3}\right)$ which gives us the two zeros and thus the two tangent line equations at the origin. The two smaller sectors have an angle measure of $\frac{\pi}{6}$ and therefore area of $\frac{\pi}{12}$ each for a combined area of $\frac{\pi}{6}$. The rest of the circle must have an area of $\frac{5\pi}{6}$ therefore the ratio is $\frac{1}{5}$ Answer E.

27. The xz-trace is found by setting y = 0 which creates a circle with radius $\sqrt{63}$ and therefore an area of 63π. Answer B.

28. $\frac{4}{3}\pi r^3 = 4\pi r^2$ when r = 3. $\frac{dV}{dt} = 4\pi r^2 \frac{dR}{dt} = 36\pi \frac{dr}{dt}$ Answer D.

29. Using trigonometric substuttion you get the integral $\int -\csc(\theta)\, d\theta = \ln(csc(\theta) + cot(\theta)) + C$

Using the substitution this reverts to ln $(\sqrt{1+x^2}$ + x) evaluated at 1 and 0 which equals ln $(\sqrt{2} + 1)$. Answer C.

30. The Ratio Test can show that only choice I of the given series of areas converges. Answer A.