ANSWERS

1. A
2. C
3. A
4. B
5. D
6. D
7. D
8. A
9. B
10. D
11. E
12. B
13. B
14. C
15. C
16. A
17. C
18. D
19. D
20. E
21. A
22. C
23. A
24. B
25. C
26. E
27. D
28. A
29. D
30. A

SOLUTIONS

1. Separating variables gives $\sec^2 y\, dy = \cos x\, dx,$ and then integrating yields $\tan y = \sin x + C.$ The initial value implies $C = 1$ so the solution is $\tan y = \sin x + 1.$

2. Separating variables gives $y\, dy = dx / x,$ and then integrating and multiplying by 2 yields $y^2 = 2\ln x + C.$ The initial value implies $C = 1$ so the solution is $y^2 = 2\ln x + 1.$ Since the initial value is positive, when we take the square root we reject the negative root to obtain a final answer of $y = \sqrt{2\ln x + 1}.$

3. Factor out P and $1/2000$ to write $P' = P(100 - P)/2000.$ Note that this differential equation is logistic with carrying capacity 100. Logistic populations grow fastest at half the carrying capacity; hence, $P = 50.$

4. The order of a differential equation refers to its highest derivative; this is 3.

5. We do not solve this equation; rather, we consider what it says. We want a function y with the property that the second derivative and y are equal. Checking the three possible choices, we find that only II and III have this property.

6. *Solution 1.* Writing $y = xy' + \dfrac{1}{y'}$ makes the possible solutions easier to verify.
For (I), we have $y' = 1$ which works. For (II), we have $y' = 1/3$ which also works. For (III) we have $y' = 1/\sqrt{x}$; plugging this into the right-hand side yields $x/\sqrt{x} + \sqrt{x} = \sqrt{x} + \sqrt{x} = 2\sqrt{x} = y,$ so this works also. By the previous work, we see that (IV) works. Hence, they all are solutions.

Solution 2. Note that the equation $y = xy' + 1/y'$ is of the form $y = xy' + g(y'),$ where $g(y')$ is a function only in $y'.$ Such an equation is a Clairaut equation; solutions are given by replacing y' with C. So the solutions are of the form $y = Cx + 1/C,$ which both (I) and (II) are. The envelope of the family of lines $y = Cx + 1/C$ is also a solution; it's envelope is $y^2 = 2x,$ and this includes (III) and (IV).

7. Let the height of the bridge be B. Then the stone's position H is $H(t) = -16t^2 + B$ since the initial velocity is zero. Setting $H(3) = -144 + B = 0$ gives $B = 144$ feet.

8. Note that $\dfrac{d}{dx}\left(\sqrt{2x}\right) = \dfrac{1}{\sqrt{2x}} = \dfrac{1}{y}.$ Thus the answer is A.

9. Using a step size of 0.1 starting at $t = 0$ and going to $t = 3$ means that we must, on the 10th step, use $t = 1$, which makes the slope undefined.

10. Separating variables results in $\cot y\, dy = -\tan x\, dx$. Integrating gives $\ln|\sin y| = \ln|\cos x| + C$. Exponentiating both sides yields $\sin y = C\cos x$.

11. Let the initial population of the bacteria be B. Malthus' Law states that $P(t) = Be^{kt}$, for constant k. With the value $P(2) = 3B$, we find that $3 = e^{2k}$, or $k = (\ln 3)/2$. Thus the equation for the bacteria is $P(t) = Be^{t(\ln 3)/2} = B \cdot 3^{t/2}$. To find the time at which the population is 100 times B, we set $100 = 3^{t/2}$. Now, since $3^4 < 100 < 3^5$, we must have $4 < t/2 < 5$, or $8 < t < 10$. Hence, the population is 100 times greater sometime between 8 PM and 10 PM.

12. When $y = 5$, we have $dy/dx = 0$. So the slope field will show a horizontal asymptote at $y = 5$. (The solution to the differential equation is $y = 5 - Ce^{-x}$, which does indeed have a horizontal asymptote at $y = 5$.)

13. *Solution 1.* For (I), we have $y' = 3$ which works. For (II), we have $y' = 75$ which does not work: $75x - 2(25)^{3/2} = 75x - 250$, not $75x - 50$. For (III) we have $y' = 768$ which works: $768x - 2(256)^{3/2} = 768x - 2(16)^3 = 768x - 8192$. Finally, for (IV) we have $y' = 3x^2$ which works: $x \cdot 3x^2 - 2(x^2)^{3/2} = 3x^3 - 2x^3 = x^3$.

Solution 2. Note that the differential equation is of the form $y = xy' + g(y')$, where $g(y')$ is a function only in y'. So this is also a Clairaut equation; solutions are given by replacing y' with C. So the solutions are of the form $y = Cx + 2(C/3)^{3/2}$, which both (I) and (III) are. The envelope of the family of lines $y = Cx + 2(C/3)^{3/2}$ is also a solution; the envelope is $y = x^3$.

14. The exponential decay of 10 grams of thorium is given by $T(t) = 10e^{kt}$ for constant k. $T(1) = 8$ implies $k = \ln(4/5)$ so that $T(t) = 10e^{t\ln(4/5)} = 10(4/5)^t$. Therefore $T(3) = 10(4/5)^3 = 128/25$. The closest integer to $128/25$ is 5.

15. By the Fundamental Theorem of Calculus, $y(3) - y(1) = \int_0^3 |2x - 3|\, dx$. The definite integral is computed by using geometry; the value is 4.5. Thus $y(3) = y(0) + \int_0^3 |2x - 3|\, dx = 5 + 4.5 = 9.5$.

16. Set $y' = 1$ and solve $1 = x - y$ to get $y = x - 1$.

17. The left-hand side is the implicit derivative of x^2y, so we may write

$$\begin{aligned}\frac{d}{dx}\left(x^2y\right) &= 1\\ \int \frac{d}{dx}\left(x^2y\right)dx &= \int dx\\ x^2y &= x + C\\ y &= \frac{1}{x}+\frac{C}{x^2}.\end{aligned}$$

The initial value implies $\frac{1}{2}+\frac{C}{4}=2\left(1+C\right)$ so that $C=-\frac{6}{7}$. Hence, $y=\frac{1}{x}-\frac{6}{7x^2}$. Finally, $y(3)=\frac{1}{3}-\frac{6}{63}=\frac{5}{21}$.

18. Once we separate variables and attempt to integrate, we are faced with $\int \sin\sqrt{x}\,dx$, for which we must use a substitution. Let $t=\sqrt{x}$ so that $t^2=x$ and $2t\,dt=dx$. Then the integral becomes $\int 2t\sin t\,dt$, which must be evaluated using parts: $u=2t$ and $dv=\sin t\,dt$, so then $du=2\,dt$ and $v=-\cos t$. Finally we have

$$y=\int \sin\sqrt{x}\,dx=\int 2t\sin t\,dt=-2t\cos t+2\sin t+C=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C.$$

At the initial value, we get $C=-2$ so that $y\left(\pi^2\right)=2\pi-2$.

19. Let's take derivatives of $y=C_1x+C_2e^x$. We get $y'=C_1+C_2e^x$ and $y''=C_2e^x$. Substituting y'' for C_2e^x in the first derivative gives us $y'=C_1+y''$, or $C_1=y'-y''$. Substituting this into the original equation gives $y=\left(y'-y''\right)x+y''=xy'+(1-x)y''$.

20. Let $S(t)$ be the amount of salt, in pounds, in the tank at time t. Salt enters the tank at 2 lbs/gal × 5 gal/min = 10 lbs/min. Since the amount of brine in the tank at time t is $100+5t-4t=100+t$, salt leaves the tank at a rate of $\frac{S}{100+t}$ lbs/gal × 4 gal/min = $\frac{4S}{100+t}$ lbs/min. So the rate of change of $S(t)$ in pounds per minute is given by $\frac{dS}{dt}=10-\frac{4S}{100+t}$. To solve this differential equation, we write it as $\frac{dS}{dt}+\frac{4S}{100+t}=10$ and use an integrating factor. That integrating factor is

$$\exp\left(\int \frac{4}{100+t}\,dt\right)=\exp\left(4\ln(100+t)\right)=(100+t)^4.$$

Then the differential equation is

$$(100+t)^4\frac{dS}{dt}+4(100+t)^3S=10(100+t)^4.$$

The left-hand side is now the derivative of $(100+t)^4S,$ so upon integrating we get $(100+t)^4S=2(100+t)^5+C.$ Solving for S yields $S(t)=2(100+t)+C(100+t)^{-4}.$ Using the initial value $S(0)=50$ gives $C=-150\times 100^4.$ Finally, we have the equation of the amount of salt:

$$S(t)=2(100+t)-150\left(\frac{100}{100+t}\right)^4.$$

But the question asks for the concentration of salt; this is $S(t)$ divided by $100+t$:

$$\text{concentration of salt}=\frac{S(t)}{100+t}=2-\frac{150\times 100^4}{(100+t)^5}.$$

So the concentration of salt 100 minutes after the process begins is

$$\frac{S(100)}{100+t}=2-\frac{150\cdot 100^4}{200^5}=2-\frac{3}{2^6}=\frac{2^7-3}{2^6}=\frac{125}{64}.$$

21. *Solution 1.* We write the differential equation as $y'-y/x=-2/x^2$ and find that the integrating factor is $\exp\left(\int 1/x\right)=\exp(-\ln x)=1/x$. Multiplying the equation by $1/x$ gives the left-hand side as the derivative of y/x and the right-hand side as $-2/x^3$. Integrating both sides results in $y/x=1/x^2+C,$ or $y=1/x+Cx.$ The initial value implies $C=6,$ and hence $y=1/x+6x.$ Therefore, $y(0.5)=2+3=5.$

Solution 2. Differentiate the equation to get $y'=y'+xy''-2/x^2.$ Then $y''=2/x^3.$ Integrating this gives $y'=-1/x^2+C.$ Now substitute this into the given differential equation:

$$y=x\left(-\frac{1}{x^2}+C\right)+\frac{2}{x}=-\frac{1}{x}+Cx+\frac{2}{x}=\frac{1}{x}+Cx.$$

Now the initial condition and the answer follow as in Solution 1.

22. Differentiate the family of curves to obtain $y+xy'=C.$ Then the differential equation for this family is $xy=\left(y+xy'\right)x-1=xy+x^2y'-1,$ or $y'=1/x^2.$ The orthogonal slope is therefore $-x^2.$ This gives a new differential equation $y'=-x^2$ which is easily solved to give $y=-x^3/3+K,$ or $3y+x^3=K.$

23. Note that $2y'y''$ is the derivative of $\left(y'\right)^2$ so integrating the differential equation yields $\left(y'\right)^2 = x + C$, or $y' = \sqrt{x+C}$. The initial value of the derivative of *y* implies $C = 3$. Now integrating $y' = \sqrt{x+3}$ results in $y = \frac{2}{3}(x+3)^{3/2} + K$. The initial value of *y* implies $K = -1/3$. Thus, $y = \frac{2}{3}(x+3)^{3/2} - \frac{1}{3}$.

24. According to Sir Isaac Newton, if *T* is the temperature and *t* is time, then $dT/dt = -k(T-10)$, where *k* is a constant. Separating variables and integrating gives $\ln\left|T-10\right| = -kt + B$. Solving this for *T* results in $T(t) = 10 + Ce^{-kt}$. The initial value $T(0) = 70$ gives $C = 60$, so that $T(t) = 10 + 60e^{-kt}$. The value $T(3) = 25$ gives $k = (\ln 4)/3$. Therefore

$$T(t) = 10 + 60\exp\left(-\frac{1}{3}t\ln 4\right) = 10 + 60\exp\left(\ln 4^{-t/3}\right) = 10 + 60\cdot 4^{-t/3}.$$

Finally, $T(9) = 10 + 60\cdot 4^{-3} = 10 + 60/64 \approx 11$.

25. Separating the variables gives $dr/r = \sec\theta\, d\theta$, and integrating gives $\ln r = \ln\left|\sec\theta + \tan\theta\right| + B$. Thus, $r = C\left|\sec\theta + \tan\theta\right|$, and the initial value implies $C = 5$. Hence,

$$r\left(\frac{5\pi}{4}\right) = 5\left|\sec\frac{5\pi}{4} + \tan\frac{5\pi}{4}\right| = 5\left|-\sqrt{2}+1\right| = 5\left(\sqrt{2}-1\right).$$

26. Let the constant deceleration be $-a$. Let $t = 0$ be the start of the braking time, and let $t = s$ be the stop time. Then $v(0) = 88, v(s) = 0, p(0) = 0,$ and $p(s) = 242$. Now since $v(t) = -at + C$, the initial value of the velocity implies $v(t) = -at + 88$. Then $p(t) = -at^2/2 + 88t + K$, and the initial value of the position implies $p(t) = -at^2/2 + 88t$. At the stop time, we have $v(s) = -as + 88 = 0$, so that $s = 88/a$. Finally, at the stop time, we have

$$p(s) = 242 = -\frac{1}{2}as^2 + 88s = -\frac{1}{2}a\left(\frac{88}{a}\right)^2 + 88\cdot\frac{88}{a} = -\frac{88^2}{2a} + \frac{88^2}{a} = \frac{88^2}{2a}.$$

Solving for *a*, we see that the constant deceleration is 16 feet per second per second.

27. A differential equation of the form $\frac{dy}{dx} = -\frac{y-k}{x-h}$ defines a hyperbola with center (h,k). However, since we are told that the solution passes through (4, 6) – which is the center! – this differential equation defines a degenerate hyperbola. That is, this defines the pair of lines $x = 4$

and $y=6$. The solution to the differential equation is therefore $(x-4)(y-6)=0$, and all the statements are true.

28. First, note that $y=2$ is a solution. Now, the characteristic equation is $m^2+5m+6=0$ whose solutions are $m=-2$ and $m=-3$. Thus the solution to the differential equation is of the form $y=C_1e^{-2x}+C_2e^{-3x}+2$. The values $y(0)=4$ and $y'(0)=0$ imply that we have the system of equations below.

$$\begin{cases} C_1+C_2=4 \\ -2C_1-3C_2=0 \end{cases} \quad \text{which implies } C_1=6, C_2=-4.$$

Hence, $y=6e^{-2x}-4e^{-3x}+2$, and $y(-\ln 5)=6\cdot 25-4\cdot 125+2=-348$.

29. Note that the differential equation is of the form $y=xy'+g(y')$, where $g(y')$ is a function only in y'. Such an equation is a Clairaut equation; solutions are given by replacing y' with C. So the solutions are of the form $y=Cx+C^2+C$. The value $y(1)=-1$ implies $-1=2C+C^2$ from which $C=-1$. Hence, $y=-x+(-1)^2+(-1)=-x$, and $y(-4)=4$.

30. To find the Taylor series solution, we first find successive derivatives of y using the given differential equation. The initial value is used to compute values of these derivatives. The first two derivatives are $y'=y^2-xy$ and $y''=2yy'-y-xy'$. The values of these derivatives at (0, 2) are $y'(0)=4-0=4$ and $y''(0)=16-2-0=14$. Therefore the first three nonzero terms of the Taylor series solution are

$$y=y(0)+y'(0)x+\frac{y''(0)}{2!}x^2=2+4x+7x^2.$$