Mu Everything Encompassing e Solutions
MAO National Convention 2015

Answers:

1. **B**	7. **D**	13. **A**	19. **B**	25. **D**
2. **D**	8. **D**	14. **B**	20. **B**	26. **B**
3. **E**	9. **A**	15. **B**	21. **A**	27. **C**
4. **D**	10. **A**	16. **B**	22. **C**	28. **B**
5. **C**	11. **E**	17. **C**	23. **D**	29. **E**
6. **C**	12. **C**	18. **C**	24. **A**	30. **C**

Solutions:

1. e is usually referred to as Euler's number, he chose the symbol e. **B**

2. The discovery of e is credited to Bernoulli while attempting to model continuously compounded interest. **D**

3. All are equal to e. A and B are common limit definitions, C is the Maclaurin series representation, and D, when evaluated $\left(\cosh x = \frac{e^x+e^{-x}}{2}, \sinh x = \frac{e^x-e^{-x}}{2}\right)$, is e. **E**

4. Move everything to one side and multiply by e^x to get $e^{2x} - 7e^x + 12 = 0$. This factors to give $(e^x - 4)(e^x - 3) = 0$, therefore $x = \ln 4, \ln 3$. Sum these to get $\ln 12$. **D**

5. The formal definition of $f(x) = \ln x$ is the area under the curve $y = \frac{1}{x}$ from 1 to x. We are given area and asked for the x; therefore, this function is the inverse of $f(x) = \ln x$ which is $g(x) = e^x$. **C**

6. i. converges by ratio test. ii. does not converge because it is a geometric series with ratio > 1. iii. is e times the alternating harmonic series which converges. iv. converges by ratio. v.does not converge by ratio test. vi. does not converge by integral test **C**

7. Using ratio test: $\lim_{n\to\infty} \frac{(x^{n+1}2^{n+1}(n+1)^{n+1})}{(3^{n+1}(n+1)!)} \cdot \frac{3^n n!}{x^n 2^n n^n} = \lim_{n\to\infty} \left(\frac{2x}{3}\right)\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{2x}{3}\right)\left(1+\frac{1}{n}\right)^n$ Notice that the second part equals e. We have $\left|\left(\frac{2x}{3}\right)e\right| < 1 \Rightarrow |x| < \frac{3}{2e}$. **D**

8. Take the derivative and set equal to 0: $\ln f = \frac{1}{x}\ln x \Rightarrow f' = f\left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right) = 0$. Solving gives $x = e$ as the only solution, with f increasing beforehand and decreasing afterwards. Therefore, the answer is $f(e) = e^{\frac{1}{e}}$. **D**

9. Using the equation for continuous interest, we have $3P = Pe^{Rt}$. Cancel the P's and take the natural logarithm of both sides to get $\ln 3 = Rt \implies t = \frac{\ln 3}{R}$. **A**

10. Convert the sum into the form of $f(a + i\Delta x)\Delta x$, where $\Delta x = \frac{b-a}{n}$. Multiply and divide the bottom by n to get $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{3}{1+\frac{i}{n}}$. Clearly, our $f(x) = \frac{3}{x}$, $a = 1$, and $b = 2$. This Riemann sum becomes the integral $\int_1^2 \frac{3}{x} dx = 3\ln 2$. **A**

11. With $u = \ln(ln\, x)$, $du = \frac{1}{\ln x} \cdot \frac{1}{x}$, our integral is now $\int_1^{\infty} u^{-3/2}\, du = -2\left(\lim_{a\to\infty} a^{-\frac{1}{2}} - 1^{-\frac{1}{2}}\right) = 2$. **E**

12. Using the identity $\cosh^2 t - \sinh^2 t = 1$, $\sinh t = \sqrt{3}$. $\sinh 2t = 2\sinh t \cosh t = 4\sqrt{3}$. Alternatively, this can be solved by knowing the expressions for the hyperbolic trig functions and doing some manipulations. **C**

13. $= \lim_{n\to\infty} \frac{1}{n}\left[\ln\frac{1}{2n} + \ln\frac{2}{2n} + \ln\frac{3}{2n} + \cdots + \ln\frac{n}{2n}\right] \implies \lim_{n\to\infty} \sum_{i=1}^{n} 2\frac{(.5)}{n} \ln\left(\frac{.5i}{n}\right) \implies$ $2\int_0^{.5} \ln x\, dx = 2(.5\ln .5 - .5) - (0) = \ln .5 - 1 = -\ln 2 - 1$. **A**

14. Using the formula for exponential growth/decay, we have $80 = 100e^{r(1)} \implies r = \ln 4/5$. To find half-life, we solve $.5 = e^{(\ln\frac{4}{5})(t)} \implies t = \frac{\ln\frac{1}{2}}{\ln\frac{4}{5}} = \frac{\ln 2}{\ln 1.25} = \log_{1.25} 2$. **B**

15. Solving the differential equation, we get $\frac{dy}{y} = dx \implies \ln|y| = x + C \implies y = \pm Ce^{x}$. Plugging in the point given (0, 0), we find that $C = 0$, so $y = 0$ is the only function that satisfies the conditions. **B**

16. $V = \pi \int_{-\infty}^{0} (e^x)^2 dx = \frac{\pi}{2}\left(e^{2(0)} - \lim_{a\to\infty} e^{2(-\infty)}\right) = \pi/2$ **B**

17. $e \approx 2.71828182845904$ The 10th digit after the decimal point is 4. There are several e mnemonics that can assist in memorizing the digits **C**

18. We have $\frac{dP(t)}{dt} = kP(t)\left(1 - \frac{P(t)}{5000}\right) \implies P(t) = \frac{5000}{1+Ce^{-kt}}$. Plugging in the two givens, we get $C = 4, -k = \frac{1}{2}\ln\frac{1}{6}$. Therefore, $P(4) = \frac{5000}{1+4\left(\frac{1}{6}\right)^2} = \frac{5000}{1+\frac{1}{9}} = 4500$. **C**

19. Maximize the derivative: $\frac{d^2P(t)}{dt^2} = k - \frac{2kP(t)}{5000} = 0 \implies P(t) = 2500$. The derivative is concave down, so this is a max. **B**

20. Take the natural log of both sides to get $\ln L = \lim_{x\to\infty} \frac{\ln(1+2e^x)}{x}$. Apply L'Hospital's rule to get $\ln L = \lim_{x\to\infty} \frac{\frac{2e^x}{1+2e^x}}{1}$. The ratio of the coefficients is 1, giving us $\ln L = 1 \implies L = e$. **B**

21. Using Newton's second law, we have $mg - kv = ma \implies 50 - kv = 5\frac{dv}{dt}$. Solve the differential equation to get $v = \frac{1}{k}\left(50 - Ce^{-\frac{kt}{5}}\right)$. Plug in $v(0)$ to get $C = 50$, and plug in $v(\ln 2)$ to get $k = 5$. Therefore, $v(\ln 5) = \frac{1}{5}\left(50 - 50e^{-\frac{5}{5}\ln 5}\right) = \frac{1}{5}(50 - 10) = 8$. **A**

22. Derive y with respect to x and plug in to get $k^3e^{kx} - k^2e^{kx} - 2ke^{kx} + 2e^{kx} = 0$. e^{kx} cannot equal 0, so cancel and solve the cubic. $k = \pm\sqrt{2}, 1$. The product of these is -2. **C**

23. Change from rectangular to polar form: $2 + 2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(e^{i\frac{\pi}{4}}\right) = e^{3\ln\sqrt{2} + i\frac{\pi}{4}}$. Therefore, $ab = \frac{3\pi}{4}\ln\sqrt{2}$. **D**

24. Expand e^{x^2} to get $f(x) = \frac{\left(1+x^2+\frac{x^4}{2!}+\cdots\right)-x^2-1}{4x^4} = \frac{\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots}{4x^4} = \frac{1}{8} + \frac{x^2}{4(3!)} + \cdots$. $\lim_{x\to 0} f(x) = \frac{1}{8}$. **A**

25. $e^{\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}} = \sum_{i=0}^{\infty} \frac{1}{i!}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}^i$. It can be shown by induction that $\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}^i = \begin{bmatrix}3^i & 0\\ 0 & 4^i\end{bmatrix}$. Therefore, we have $e^{\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}} = \begin{bmatrix}\sum_{i=0}^{\infty}\frac{3^i}{i!} & 0\\ 0 & \sum_{i=0}^{\infty}\frac{4^i}{i!}\end{bmatrix} = \begin{bmatrix}e^3 & 0\\ 0 & e^4\end{bmatrix}$. **D**

26. Factor out a $\frac{1}{e}$ and multiply the top and bottom by e^{2x} to get $\frac{1}{e}\int_0^\infty \frac{e^{2x}dx}{e^{4x}+1}$. Using a u sub of $u = e^{2x}, \frac{du}{2} = e^{2x}$, we get $\frac{1}{2e}\int_1^\infty \frac{du}{u^2+1} = \frac{1}{2e}(\lim_{a\to\infty} \arctan a - \arctan 1) = \frac{1}{2e}\left(\frac{\pi}{4}\right)$. **B**

27. Plug in the Maclaurin series for e^x. We are looking for the x^{10} coefficient times 10!, because all others will become 0 when $f^{(10)}(0)$ is evaluated. This is $x^2\left(\frac{x^8}{8!}\right) + 1\left(\frac{x^{10}}{10!}\right) = \frac{91}{10!}x^{10}$. **C**

28. The probability of loosing is $\left(1 - \frac{1}{1{,}000{,}000}\right)$. The probability of loosing 1,000,000 times in a row is $\left(1 - \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ which is close to the limit definition of e^x. Therefore, the probability is approximately $1/e$. **B**

29. This sum equals $\sum_{n=0}^{\infty} \frac{e^{ni\theta}}{3^n} = Re\left[\sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{3}\right)^n\right]$ which is an infinite geometry series. We can sum this as $\frac{1}{1-\frac{e^{i\theta}}{3}} = \frac{3}{3-\cos\theta - i\sin\theta}$. Multiply by the conjugate to isolate the real part and get $Re\left[\frac{3(3-\cos\theta + i\sin\theta)}{9-6\cos\theta+\cos^2\theta+\sin^2\theta}\right] = \frac{9-3\cos\theta}{10-6\cos\theta}$. Plugging in the given value, we get $\frac{9-3\left(\frac{1}{3}\right)}{10-6\left(\frac{1}{3}\right)} = 1$. **E**

30. We can solve this by using Euler's formula. $i^{i^3} = i^{-i} = \left(e^{i\left(\frac{\pi}{2}\right)}\right)^{-i} = e^{\frac{\pi}{2}}$. **C**