You have 60 minutes to complete this 30-question test. The answer choice E. NOTA denotes that "None of These Answers" is correct. Good luck!

1. Find the length of the shortest first-quadrant line segment with endpoints on the coordinate axes that is tangent to the graph of $y = \frac{1}{x}$.

A) $\sqrt{2}$ B) 2 C) $2\sqrt{2}$ D) $4\sqrt{2}$ E) NOTA

2. The water level in Buchholz Bay varies sinusoidally. At high tide today, when the Buchholz team started to practice at 8 AM, the water level was 15 feet. At low tide, when they finished practice 6 hours later at 2 PM, the water level was 3 feet. How fast, in feet per hour, was the water level dropping during ciphering practice – that is, at noon – today?

A) 3 B) $\frac{\pi\sqrt{3}}{2}$ C) $3\sqrt{3}$ D) $\frac{\pi\sqrt{2}}{2}$ E) NOTA

3. Which of the following statements is/are true?
I. If $f(x)$ is continuous at $x = c$, then $f'(c)$ exists.
II. If $f'(c) = 0$, then f has a local maximum or minimum at $(c, f(c))$.
III. If $f''(c) = 0$, then the graph of f has an inflection point at $(c, f(c))$.
IV. If f is differentiable at $x = c$, then f is continuous at $x = c$.
V. If f is continuous on (a, b) and $f(a) = f(b)$, then f attains a maximum value on (a, b).

A) All are true B) II, III, IV, V only C) II, III only D) IV only E) NOTA

4. The American League East division consists of 5 teams: the New York Yankees, the Boston Red Sox, the Tampa Bay Rays, the Baltimore Orioles, and the Toronto Blue Jays. If each of these teams plays each of the other teams in the division 15 times during the regular season, how many intra-division games (games in which both teams are from this division) will occur during a season?

A) 75 B) 150 C) 300 D) 450 E) NOTA

5. If $f(x) = 2x^3 - 5x^2 + 7x - 1$, and $g(x) = f^{-1}(x)$, find the value of the reciprocal of $g'(3)$.

A) 1/3 B) 2/3 C) 3/2 D) 2 E) NOTA

6. The rate of change of the volume of Mr. Smith's head – which happens to be a perfect sphere – when receiving compliments is 4π units cubed/sec. Give the rate of change of the surface area (in units squared/sec) of Mr. Smith's head at the time when the radius of his head is 2.

A) ¼ B) $\frac{\pi}{2}$ C) π D) 4π E) NOTA

7. Evaluate the following interval to show your worth in both pre-calc and calc:

$$\int_0^2 \frac{x\,dx}{\sqrt{4 - x^2}}$$

A) $\sqrt{3}$ B) $2\sqrt{2}$ C) $2\sqrt{3}$ D) 2 E) NOTA

8. A giant asteroid in the shape of a 2D plane (an admittedly unrealistic structure) has struck the Earth! Assume that Earth is a sphere. The actual number for the radius of the Earth is too large to deal with, so we'll call it *r*. The asteroid divided Earth into two parts at distance *h* (0<*h*<*r*) from the center, creating two distinct planets, *Eartha Major* (the larger part) and *Eartha Minor* (the smaller part). Give the volume of *Eartha Minor*.

A) $\frac{\pi}{3}(2r^3 + 3r^2h - h^3)$ B) $\frac{\pi}{3}(2r^3 + h^3 - 3r^2h)$ C) $\frac{\pi h}{3}(3r^2 - h^2)$
D) $\frac{4}{3}\pi r^3 - r^2h$ E) NOTA

9. Going back to your Conics & Coordinate Geometry days, but with an added twist…
The general solution of the differential equation $\frac{dy}{dx} = \frac{1-2x}{y}$ is a family of which of the following types?

A) Circles B) Hyperbolas C) Parabolas D) Ellipses E) NOTA

10. Which of the following does not satisfy the Mean Value Theorem on the given interval?

A) $f(x) = x^2 - 2x$ on $[-3, 1]$ B) $f(x) = e^x$ on $[-1, 1]$ C) $f(x) = x + \frac{1}{x}$ on $[-1, 1]$
D) $f(x) = x^{2/3}$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$ E) NOTA

11. Evaluate $\lim_{n\to\infty} \frac{2}{n}\left(\ln\left(1+\frac{2}{n}\right) + \ln\left(1+\frac{4}{n}\right) + \ln\left(1+\frac{6}{n}\right) + \ldots \ln\left(1+\frac{2n}{n}\right)\right)$

A) 3ln3 – 2 B) $e^3 - 1$ C) 2ln2 D) 0 E) NOTA

12. Find the sum of the coefficients of the terms of the 4th degree Maclaurin polynomial for:
$f(x) = 3e^{2x}$.

A) 10 B) 16 C) 21 D) 24 E) NOTA

13. The position (x, y) of a particle at time t is given parametrically by $x = t^2$ and $y = \frac{t^3}{3} - t$.
Find the distance the particle travels between $t = 1$ and $t = 2$.

A) 4/3 B) 7/3 C) 4 D) 10/3 E) NOTA

14. Evaluate the following integral, if possible:

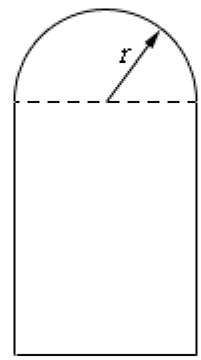
$$\int_2^4 \frac{dx}{(x-3)^3} = ?$$

A) 2 B) -2 C) 0 D) 2/3 E) NOTA

15. What is the value of $A''(x)$ at $x = 0$, if $A(x) = \frac{x^2-3x+2}{x^3-6x^2+11x-6}$?

A) $-\frac{1}{9}$ B) $\frac{2}{9}$ C) $-\frac{2}{27}$ D) -1 E) NOTA

16. Doker is designing his new apartment. He needs to build a window that overlooks Central Park. He has two specifications. He wants the window to be shaped as shown, with the bottom as a rectangle and the top as a semicircle. He also wants the window to let in the most light (have the biggest area) possible while only using 12 meters of window framing material (the perimeter cannot exceed 12). What should be the radius (shown as r) of the semicircle?



A) $\frac{12}{\pi}$ B) $\frac{24}{4+\pi}$ C) $\frac{24-4\pi r-\pi^2 r}{4+\pi}$ D) $\frac{12}{4+\pi}$ E) NOTA

17. Evaluate the indefinite integral:

$$\int \frac{x-6}{x^2-3x}\,dx$$

A) $\ln|x^2(x-3)|+C$ B) $-\ln|x^2(x-3)|+C$ C) $\ln|x^2/(x-3)|+C$
D) $\ln|(x-3)/x^2|+C$ E) NOTA

18. Please evaluate the following limit:

$$\lim_{x\to\infty} x^{1/x}$$

A) 0 B) 1 C) e D) ∞ E) NOTA

19. Everyone loves some good physics applications! A ball is dropped from an 80 m tall building. How long does the ball take to reach the ground? For this problem, use that the acceleration due to gravity is -10 m/s^2.

A) 8 seconds B) 16 seconds C) 8.9 seconds D) 4 seconds E) NOTA

20. In a certain region, the electric field varies with the radius away from the origin by the equation $E = -6r^2 + 4r + 3$, where r is given in meters and E is N/C. Given that the potential difference can be calculated using the equation that relates the potential difference to the electric field: $\Delta V = -\int E\,dr$, where the limits of integration relate the distance between the two points of interest, find the potential difference between the origin and the point (3, 4).

A) 315 V B) 185 V C) 64 V D) -165 V E) NOTA

21. Find the interval of convergence of the series: $\sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{(n+1)^2}$.

A) (1, 3) B) [1, 3) C) [1, 3] D) [3, ∞) E) NOTA

22. The base of a solid is the region bounded by $y = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$. Each cross section perpendicular to the x-axis is a square. The volume of the solid is:

A) $e^2/2$ B) $e^2 - 1$ C) $1 - e^{-2}$ D) $\frac{1}{2} - \frac{1}{2}e^{-2}$ E) NOTA

23. The average value of $f(x) = 3 + |x|$ on the interval $[-2, 4]$ is:

A) 14/3 B) 8/3 C) 16/3 D) 20/3 E) NOTA

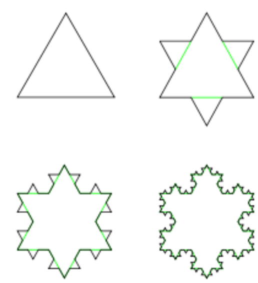
24. The Frazer Function – denoted $\aleph(x)$ – has the interesting property that $\aleph''(x) = \aleph(x)$. Given only this information, which of the following could be the Frazer function?

I. $\aleph(x) = e^x$ II. $\aleph(x) = \sinh x$ III. $\aleph(x) = \frac{e^x + e^{-x}}{2}$ IV. $\aleph(x) = 0$

A) IV only B) I, IV only C) I, II, IV only D) I, III, IV only E) NOTA

25. The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment (side of the triangle) as follows:
1) divide the line segment into three segments of equal length.
2) draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
3) remove the line segment that is the base of the triangle from step 2.

After one iteration of this process, the resulting shape is the outline of a hexagram. The first four iterations are shown to the right. The Koch snowflake is the limit approached as the above steps are followed over and over again.



Your goal in this question is to find the area of the Koch snowflake in terms of a_0, the area of the original triangle, after an infinite number of iterations. (As an extra challenge after this test, see if you can find the perimeter of such a figure!)
Here are some hints to get you started:

- In each iteration a new triangle is added on each side of the previous iteration, so the number of new triangles added in iteration *n* is: $T_n = N_{n-1} = 3 * 4^{n-1} = \frac{3}{4} * 4^n$.
- The area of each new triangle added in an iteration is one ninth of the area of each triangle added in the previous iteration, so the area of each triangle added in iteration *n* is: $a_n = \frac{a_{n-1}}{9} = \frac{a_0}{9^n}$, where a_0 is the area of the original triangle.

A) $\frac{3}{2}a_0$ B) $\frac{10}{7}a_0$ C) $\frac{7}{2}a_0$ D) $\frac{8}{5}a_0$ E) NOTA

26. The length of $x = e^t \cos t$, $y = e^t \sin t$ from $t = 2$ to $t = 3$ is equal to:

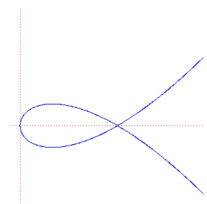
A) $\sqrt{2}e^2\sqrt{e^2 - 1}$ B) $\sqrt{2}(e^3 - e^2)$ C) $2(e^3 - e^2)$
D) $e^3(\cos 3 + \sin 3) - e^2(\cos 2 + \sin 2)$ E) NOTA

27. Another quick integration question, for those of you who know your stuff cold (*a* is positive):

$$\int_0^\infty a^{-x/2}\,dx$$

A) $\frac{2}{\ln a}$ B) $2a$ C) 2 D) ∞ E) NOTA

28. Pikachu's Cubic is given by the Cartesian equation: $3ay^2 = x(x-a)^2$, where a is a constant, and is also shown to the right. Note that although the picture will change depending on the value of a, most of the important properties you may find helpful remain unchanged. In terms of a, and assuming only positive values for a, find the y-coordinate of the point that represents the only local maximum of this curve (notice that there is no global maximum).



A) $\frac{4}{81}a^2$ B) $\frac{2}{9}a$ C) $\frac{a^{3/2}}{24}$ D) $\frac{a}{\sqrt{24}}$ E) NOTA

29. No true Princetonian will let you off without a bit of economics. Consider a firm with production function $f(K,L) = \sqrt{KL}$. (This production function specifies the maximum amount of output the firm can produce using K units of capital and L units of labor) The prices for each unit of capital and labor are $p_L = 1$ and $p_K = 2$, respectively. Assume that we are operating in, *and are only concerned about*, the short run, so that capital is fixed at K = 4. Also assume that each unit produced can be sold at a price of 2. Determine the amount of labor this firm should employ if it seeks to maximize its profit.

A) 0 – the firm should be shut down B) 1 C) 4 D) 9 E) NOTA

30. You have all heard of the Prisoner's Dilemma and Nash Equilibrium (if not, don't worry, no background knowledge of Game Theory is needed to solve this final question). This question extends that simple example into a more general strategy for solving for a Nash Equilibrium in a setting known as the Cournot model.

Each of two firms has the cost function $TC(y) = 30y$. This function specifies a firm's total cost of producing y units of output. The (inverse) market demand curve for the firms' output is $P = 120 - Q$, where Q is the total output produced by the two firms combined ($y_1 + y_2$), and P is the market price at which each unit of output is slod. (Note that this means that the market price for the product declines as either firm increases its output.) Our goal is to find the outputs of each of the firms in a Cournot model. To do so, we must find the "best response function" for Firm 1 and Firm 2 – that is, the profit-maximizing output for a firm, given that the other firm produces at some level. The setup for Firm 1 is provided for you:

Firm 1's profit is $y_1(120 - y_1 - y_2) - 30y_1$, where y_1 and y_2 are the outputs of Firm 1 and Firm 2, respectively (can you see why? Profit of Firm 1 = Quantity by firm 1 * Market Price – Cost of production for Firm 1). Take the derivative of this profit with respect to y_1, treating y_2 as a constant, and set it equal to zero to obtain an expression for y_1 in terms of y_2. If you did this right, you will have a function in the form of $y_1 = (A - By_2)/C$, where A, B, C are constants you must solve for. Call this equation (1).

Then repeat the process described above to get an expression for y_2 in terms of y_1. Call this equation (2). Equations (1) and (2) should be very similar – do you see why?

Equations (1) and (2) are two equations in two variables. Substitute one into the other, and solve for y_1 and y_2. Please report the strategically optimal output for Firm 2 (that is, the value you obtained for y_2).

A) 15 B) 30 C) 60 D) 120 E) NOTA