Nationals 2015: Mu Gemini Answers

1. C
2. B
3. D
4. B
5. E
6. D
7. D
8. B
9. D
10. C
11. A
12. C
13. D
14. E
15. C
16. D
17. C
18. B
19. D
20. B
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23. A
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26. B
27. A
28. B
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30. B

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1. C – For any value of x, say C, the point (C, 1/C) lies on the graph, meaning that the tangent line has the equation $y = -\frac{1}{C^2}x + \frac{2}{C}$. This line has intercepts (2C, 0) and (0, $\frac{2}{C}$). So we find the C which minimizes $\sqrt{4C^2 + \frac{4}{C^2}}$, which is done by taking the derivative and setting it equal to 0. We find C = 1, meaning that the line segment has length $\sqrt{8} = 2\sqrt{2}$.

2. B – The equation for the water level is in the form $h(t) = a\sin(bt + c) + d$, where $a, b, c,$ and d are constants. High tide occurs when $\sin(bt + c) = 1$, and low tide occurs when $\sin(bt + c) = -1$. Therefore, $a + d = 15$, and $-a + d = 3$, so $a = 6,\ b = 9$. Letting 8 AM be $t = 0$, the sine function equals 1 (high tide, $t = 0$) at $\pi/2$ and -1 (low tide, $t = 6$) at $3\pi/2$. Therefore, $\sin c = 1 \ \rightarrow c = \pi/2$ and $\sin(6b + c) = -1 \ \rightarrow b = \pi/6$. The resulting equation is thus $h(t) = 6\sin(\frac{\pi}{6}t + \frac{\pi}{2}) + 9$, and we must evaluate $h'(4)$ to get the rate of change at noon. $h'(t) = \pi\cos(\frac{\pi}{6}t + \frac{\pi}{2})$, so $h'(4) = -\frac{\pi\sqrt{3}}{2}$, meaning the water is dropping a rate of $\frac{\pi\sqrt{3}}{2}$.

3. D – To show a statement is false, you just need to find one counterexample. I is false – think of absolute value functions. II and III can be shown false by the same example: y=3 – first and second derivatives are zero at all points, but there are no local max/mins and no inflection points. IV is true by definition. V is false – this is like Rolle's Theorem, which says that there must be a point on the interval where the first derivative is zero, but this point does not have to be a maximum.

4. B – There are 5C2 = 10 possible matchups. Each matchup occurs 15 times, so 10*15 = 150.

5. E (3) – $g'(3) = \frac{1}{f'(1)} = \frac{1}{6(1*1)-10(1)+7} = \frac{1}{3}$, and the reciprocal is 3.

6. D – $4\pi r^2\frac{dr}{dt} = 4\pi$, so $\frac{dr}{dt} = \frac{1}{r^2} = \frac{1}{4}$. The surface area is $4\pi r^2$, and $8\pi r\frac{dr}{dt} = 4\pi$.

7. D – Let $x = 2\sin\theta$. Then $\sqrt{4 - x^2}$ simplifies to $2\cos\theta$, and $dx = 2\cos\theta\, d\Theta$. These cancel out, leaving only x (or $2\sin\theta$) in the integral. Don't forget to change the bounds as well: when x=0, $\theta = 0$, and when x=2, $\theta = \pi/2$. We evaluate $-2\cos\theta$ with bounds 0 and $\pi/2$, giving the answer of 2.

8. B – We get a section that is less than half of the sphere, whose volume is found by finding the volume of disks revolved about the y-axis. A picture is useful for visualization. This is evaluated as the integral:

$$\pi\int_h^r (r^2 - y^2)dy = \frac{\pi}{3}(2r^3 + h^3 - 3r^2h)$$

9. D – By separation of variables, $y\,dy = (1 - 2x)dx$. Integrating gives $\frac{1}{2}y^2 = x - x^2 + C$, or $y^2 = 2x - 2x^2 + k$, or the final form, $2x^2 + y^2 - 2x = k$, which is the equation for a family of ellipses.

10. C – For the Mean Value Theorem to hold over a given interval, the function must be continuous on the closed interval. $\frac{1}{x}$ is not continuous at 0, making function C not continuous over the interval.

11. A – $\int_1^3 \ln(x)dx = \left(x\ln(x) - x\right)\Big]_1^3 = 3\ln 3 - 3 - (0-1) =$
$3\ln 3 - 2$

12. C – Since $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$, by substitution $3e^{2x} = 3 + 6x + 6x^2 + 4x^3 + 2x^4$. Summing the coefficients, 3 + 6 + 6 + 4 + 2 = 21.

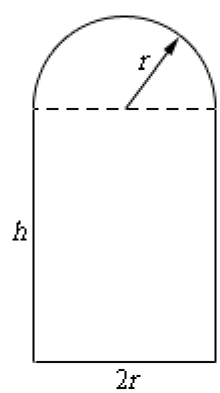
13. D – $dx = 2tdt,\ \ dy = (t^2 - 1)dt$. Therefore, $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{4t^2 + t^4 - 2t^2 + 1}\,dt$.

$$s = \int_1^2 \sqrt{(t^2+1)^2}\,dt = \frac{t^3}{3} + t\,\Big] = \frac{10}{3}.$$

14. E – This integral is equivalent to: $\int_2^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2}$. Neither of the two integrals converges, so the original integral also diverges.

15. C – This question could be very annoying if you try to use the quotient rule on the original function. The trick is to factor the top and bottom into: $\frac{(x-1)(x-2)}{(x-1)(x-2)(x-3)}$, simplifying to $A(s) = (x-3)^{-1}$. $A''(s) = \frac{2}{(x-3)^3}$, so the answer is -2/27.

16. D – Start by filling in the other dimensions in the picture, as shown:



This is then an optimization problem with a constraint:
Maximize : $A = 2hr + \frac{1}{2}\pi r^2$
Constraint : $12 = 2h + 2r + \pi r$

Solve the constraint for $h = 6 - r - \frac{1}{2}\pi r$, then plug that into the area equation: $A(r) = 2r\left(6 - r - \frac{1}{2}\pi r\right) + \frac{1}{2}\pi r^2 = 12r - 2r^2 - \frac{1}{2}\pi r^2$. $A'(r) = 12 - r(4+\pi)$, which means that the critical point occurs at $r = \frac{12}{4+\pi}$. Check the second derivative to show that this is a maximum.

17. C – Use partial fractions to evaluate this integral:

$$\frac{x-6}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$$
$$x - 6 = A(x-3) + Bx$$

Letting x=0, it is clear that A = 2, and setting x = 3 shows that B = -1.
Now we have $\int \frac{2}{x} - \frac{1}{x-3} dx = 2\ln|x| - \ln|x-3| + C$, which can be simplified to the answer shown in C.

18. B – Let $y = x^{1/x}$, then $\ln y = \frac{\ln x}{x}$. Because this fraction is in the form ∞/∞, L'Hopital's rule applies. $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1/x}{1}$, so y approaches $e^0 = 1$.

19. D – The first integration gives $v = \frac{1}{2}gt^2 + v_0$, and the second gives $y = y_0 + v_0 t + \frac{1}{2}gt^2$. We want to find the time at which $y = 0, y_0 = 80$, and $v_0 = 0$, since initial vertical velocity is always zero when an object is dropped. Plug in the known values and solve for t:

$$0 = 80 + 0 * t + \frac{1}{2}(-10)t^2 \quad \rightarrow \quad t = 4\,s$$

20. B – The limits of the integration will be from 0 to 5, because the radius is 5 for the point (3, 4). Therefore:

$$\Delta V = -\int_0^5 (-6r^2 + 4r + 3)dr = -(-250 + 50 + 15) = 185\,Volts$$

21. C –

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(x-2)^n}\right| = \lim_{n\to\infty}|x-2| = |x-2|$$

$$|x-2| < 1 \rightarrow 1 < x < 3$$

We are not yet done, since we still need to check the endpoints:

$x = 1$ gives $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ and $x = 3$ gives $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$. The first series converges by direct comparison to $\frac{1}{n^2}$ and the second one converges absolutely using the same direct comparison test. Both "endpoint series" converge, meaning that the endpoints should be included in the interval of convergence, making the answer [1, 3].

22. D – $\int_0^1 (e^{-x})^2 dx = \frac{1}{2} - \frac{1}{2}e^{-2}$

23. A – This is calculated as $\frac{1}{4-(-2)}\int_{-2}^{4}(3 + |x|)dx = \frac{1}{6}\left(\int_0^4 (3 + x)dx + \int_{-2}^{0}(3 - x)dx\right)$ = 14/3.

24. E – All 4 options could be the Frazer function. I and IV are fairly obvious – any order derivative of e^x or 0 is always itself. III is the expanded version of $\cosh x$, and its first derivative is $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, the second derivative of both II and III would equal itself.

25. D – You are given the number of triangles added in each iteration, as well as their areas. To find the amount of new area added each iteration, we simply multiply. The total new area added in iteration *n* is:

$b_n = \frac{3}{4} * 4^n * \frac{a_0}{9^n} = \frac{3}{4} * \left(\frac{4}{9}\right)^n * a_0$. The total area after *n* iterations as *n* approaches infinity, then, is $A_n = a_0 + \sum_{k=1}^{\infty} b_k = a_0\left(1 + \frac{3}{4}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k\right)$. The summation part is an infinite geometric series, which has a sum of first term / (1 – common ratio), so equates to $\frac{4/9}{1-4/9} = \frac{4}{5}$. Thus, $A_n = a_0\left(1 + \frac{3}{4} * \frac{4}{5}\right) = \frac{8}{5}a_0$.

26. B – Use the formula $\int_{t_0}^{t_1}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$ to obtain the integral:

$$\int_2^3 \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2}\, dt = \sqrt{2}e^3 - \sqrt{2}e^2.$$

27. A – This integral evaluates to $\frac{-2a^{-x/2}}{\ln a}$. Evaluated from 0 to ∞ gives 0 – (-2/ ln(a)) = $\frac{2}{\ln a}$.

28. B – First note that the curve will intersect itself at the point $(a, 0)$ – this is easy to see when plugging in $y = 0$. So we know we want to focus on the section of the curve in the first quadrant with x-coordinates less than a. It is then probably easiest to convert the equation into the form $y = \pm\sqrt{\frac{x(x-a)^2}{3a}}$. We only have to worry about the positive part, so $\frac{dy}{dx} = \frac{1}{2} * \frac{1}{3a} * (x^3 - 2ax^2 + a^2x)^{-\frac{1}{2}}(3x^2 - 4ax + a^2)$, which must equal 0 at a local max. The only way to make that equal 0 is to set $3x^2 - 4ax + a^2 = (3x - a)(x - a) = 0$. The solution $x = a$ is not feasible because it makes the denominator of the derivative equal 0, so we know the local max is at $x = a/3$. That is then plugged back into the original equation, and taking the positive value (since we're looking for a maximum), we get answer B.

29. C – Profit is equal to total revenues minus total costs, so we want to maximize: $2\sqrt{KL} - L - 2K$. With K set at 4, this becomes a simple maximization problem over one variable. Max $4\sqrt{L}$ – L – 8. Take the first derivative and set equal to 0, and solving gives L = 4. Note that this yields a negative profit (-4) for the firm. However, the firm should still operate because if it shut down, setting labor (and production) equal to 0, the profits would be -8 due to the large fixed costs, which is even worse than when the firm does operate.

30. B – Taking the derivative of the profit function given in the question with respect to y_1 gives: $120 - 2y_1 - y_2 - 30 = 0$, which is then simplified to: $y_1 = (90 - y_2)/2$. Since the cost functions for the two firms are the same (and the market price is a factor of the sum of both y_1 and y_2), the best response function for Firm 2 must be symmetrical: $y_2 = (90 - y_1)/2$. If this intuition is not obvious, it is simple to redo the steps described in the problem for y_2. Substituting one equation into the other, we obtain y_1 = (90 – (90 - y_1)/2)/2, so that y_1=30 and y_2=30. It is additionally interesting to note the logical result that the optimal outputs of the two firms are the same – this must occur when neither firm has any strategic dominance over the other.