ANSWERS

1. A
2. D
3. B
4. D
5. B
6. C
7. B
8. A
9. B
10. C
11. C
12. A
13. E
14. B
15. A
16. B
17. D
18. B
19. D
20. B
21. C
22. A
23. B
24. C
25. E
26. A
27. C
28. B
29. D
30. E

SOLUTIONS

1. $\int_1^2 v(t)\,dt = 3t^3 - 4t^2 + 5t\Big|_1^2 = 24 - 16 + 10 - 4 = 14.$

2. Since $\int_{-1}^{k} \left(3x^2 - 2x\right) dx = x^3 - x^2\Big|_{-1}^{k} = k^3 - k^2 + 2,$ we set this equal to 6 and solve. From $k^3 - k^2 - 4 = 0,$ we get $(k-2)\left(k^2 + k + 2\right) = 0$ so that the only real solution is $k = 2$.

3. Note that the area under $f(x)$ from $x = 3$ to $x = 7$ is the difference of the two given integrals; this is $-4 - 5 = -9$. Now we compute:

$$\begin{aligned}\int_7^3 \left(x + f(x)\right) dx &= \int_7^3 x\,dx + \int_7^3 f(x)\,dx \\ &= -\int_3^7 x\,dx - \int_3^7 f(x)\,dx \\ &= -\frac{1}{2}x^2\Big|_3^7 - (-9) \\ &= -20 + 9 = -11.\end{aligned}$$

4. The integral has no closed-form antiderivative; we use geometry. Since $\int_0^4 |x - 2|\,dx$ is the area of two isosceles right triangles with legs of length 2, and $\int_0^4 3\,dx = 12,$ the definite integral is equal to $2 \times 2 + 12 = 16.$

5. $\int \frac{6x^4 + 3x^2 + 1}{x^2}\,dx = \int \left(6x^2 + 3 + \frac{1}{x^2}\right) dx = 2x^3 + 3x - \frac{1}{x} + C = \frac{2x^4 + 3x^2 - 1}{x} + C.$

6. We use integration by parts, with $u = x,\ dv = e^{-x}\,dx$ so that $du = dx,\ v = -e^{x}$. Thus,

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C.$$

7. Using the substitution $u = x^2,$ the antiderivative is $\frac{1}{2}\cosh\left(x^2\right) + C.$

8. This is a standard double integral computation.

$$\begin{aligned}\int_0^1\int_1^{e^x}(x+y)\,dy\,dx &= \int_0^1\left(xy+\frac{1}{2}y^2\right)\Big|_1^{e^x}dx\\ &= \int_0^1\left(xe^x+\frac{1}{2}e^{2x}-x-\frac{1}{2}\right)dx\\ &= xe^x-e^x+\frac{1}{4}e^{2x}-\frac{1}{2}x^2-\frac{1}{2}x\Big|_0^1\\ &= e-e+\frac{1}{4}e^2-\frac{1}{2}-\frac{1}{2}-\left(-1+\frac{1}{4}\right)\\ &= \frac{1}{4}e^2-1+\frac{3}{4}=\frac{e^2-1}{4}.\end{aligned}$$

9. The integrand of an arc length integral is the square root of 1 plus the derivative squared; hence, $f'(x)=12x^2-4$. The antiderivative is $f(x)=4x^3-4x+C$. Using the point (2, 22) gives $C=-2$, so that the curve has equation $y=4x^3-4x-2$.

10. A bounded function f on a finite interval is Riemann integrable if and only if the set of discontinuities of f is a set of measure zero (for elementary calculus, that means there are finitely many discontinuities). The function in the integrand is equal to 1 for all real numbers, except at integer values, in which case, the function is 0. On the interval [0, 10], we have 11 points of discontinuity, but these don't matter! The integral is equivalent to $\int_0^{10}1\,dx=10$.

11. First, compute the definite integral.

$$\int_2^6(ax+b)\,dx=\frac{1}{2}\left(ax^2+bx\right)\Big|_2^6=18a+6b-2a-2b=16a+4b$$

Thus, we have the equation $16a+4b=8$, or $4a+b=2$. The sum of $4a$ and b is even, so both are even or both are odd. Clearly, $4a$ is even. Thus, b must be even.

12. $T_2=2\cdot\frac{a+c}{2}+2\cdot\frac{c+a}{2}=2a+2c,\ T_4=\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+c}{2}+\frac{c+a}{2}=\frac{2a+2b+4c}{2}=a+b+2c.$
Thus, $T_2-T_4=2a+2c-a-b-2c=a-b.$

13. Note that $F(x)=\int_1^x \frac{1}{t}\,dt=\ln|t|\Big|_1^x=\ln|x|$. Then the average value of $F(x)=\ln|x|$ is

$$\frac{1}{3-1/3}\int_{1/3}^{3}\ln|x|\,dx=\frac{3}{8}\left(x\ln|x|-x\right)\Big|_{1/3}^{3}=\frac{3}{8}\left(3\ln 3-3-\frac{1}{3}\ln\frac{1}{3}+\frac{1}{3}\right)=\frac{3}{8}\left(-\frac{8}{3}+\frac{10}{3}\ln 3\right)=-1+\frac{5}{4}\ln 3.$$

14. We use the substitution $u=4+5\sqrt{x}$. So when $x=1$, then $u=9$ and when $x=9$, then $u=19$. Also under this substitution, $x=(u-4)^2/25$ so that $dx=2(u-4)(du)/25$. The integral is thus

$$\int_9^{19}\frac{1}{u}\cdot\frac{2(u-4)}{25}\,du=\frac{2}{25}\int_9^{19}\left(1-\frac{4}{u}\right)du=\frac{2}{25}\left(u-4\ln|u|\right)\Big|_9^{19}=\frac{2}{25}\left(10+4\ln\frac{9}{19}\right).$$

Factoring a 2, we get the answer $\frac{4}{25}\left(5+2\ln\frac{9}{19}\right)$.

15. $\int_e^{e^2}\left(16-2x-\frac{8}{x}\right)dx=16x-x^2-8\ln|x|\Big|_e^{e^2}=16e^2-e^4-16-16e+e^2+8=17e^2-e^4-16e-8$

16. $\frac{1}{3}\int_0^3 4^{\sinh x}\cosh x\,dx=\frac{1}{3}\cdot\frac{4^{\sinh x}}{\ln 4}\Big|_0^3=\frac{1}{3}\left(\frac{4^{\sinh 3}}{\ln 4}-\frac{1}{\ln 4}\right)=\frac{4^{\sinh 3}-1}{3\ln 4}$

17. $\int_2^3\frac{1}{x^2-4x+5}\,dx=\int_2^3\frac{1}{(x-2)^2+1}\,dx=\arctan(x-2)\Big|_2^3=\arctan 1=\frac{\pi}{4}$

18. $\int_0^1 e^{e^x+x}\,dx=\int_0^1 e^{e^x}e^x\,dx=e^{e^x}\Big|_0^1=e^e-e$

19. The integral diverges when $2p-3\le 1$. Hence, $p\le 2$.

20. Note that the numerator is almost equal to $(x+1)^5$, so we add and subtract x and 1 in the numerator. This yields

$$\begin{aligned}\int_0^1 \frac{x^5+5x^4+10x^3+10x^2+5x+1-x-1}{(x+1)^5}\,dx &= \int_0^1\left(\frac{x^5+5x^4+10x^3+10x^2+5x+1}{(x+1)^5}-\frac{x+1}{(x+1)^5}\right)dx\\ &= \int_0^1\left(1-\frac{1}{(x+1)^4}\right)dx\\ &= x+\frac{1}{3(x+1)^3}\Big|_0^1\\ &= 1+\frac{1}{24}-\frac{1}{3}=\frac{17}{24}.\end{aligned}$$

21. We use the method of cylindrical shells.

$$2\pi\int_0^{\sqrt{\ln 10}} xe^{-x^2}\,dx = 2\pi\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^{\sqrt{\ln 10}} = -\pi\left(e^{-\ln 10}-1\right)=\pi\left(1-\frac{1}{10}\right)=\frac{9\pi}{10}$$

22. Use integration by parts to get the antiderivative: $u=\sec^{-1}x$ and $dv=dx$ so that $du=\frac{dx}{|x|\sqrt{x^2-1}}$ and $v=x$. Note that the interval over which we integrate is positive, so we may take $du=\frac{dx}{x\sqrt{x^2-1}}$. Therefore, we have $\int \sec^{-1}x\,dx = x\sec^{-1}x-\int\frac{1}{\sqrt{x^2-1}}\,dx$. To evaluate this second integral, we use a trigonometric substitution. Let $x=\sec\theta$ so that $dx=\sec\theta\tan\theta\,d\theta$. Then the second integral becomes

$$\int\frac{1}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta=\int\sec\theta\,d\theta=\ln\left|\sec\theta+\tan\theta\right|+C.$$

Undoing the substitution and (finally!) evaluating the definite integral gives

$$\begin{aligned}\int_{2/\sqrt{3}}^{2}\sec^{-1}x\,dx &= x\sec^{-1}x-\ln\left|x+\sqrt{x^2-1}\right|\Big|_{2/\sqrt{3}}^{2}\\ &= 2\cdot\frac{\pi}{3}-\ln\left|2+\sqrt{3}\right|-\frac{2}{\sqrt{3}}\cdot\frac{\pi}{6}+\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right|\\ &= \frac{2\pi}{3}-\frac{\pi}{3\sqrt{3}}+\ln\left|\frac{\sqrt{3}}{2+\sqrt{3}}\right|\\ &= \frac{2\pi}{3}-\frac{\pi}{3\sqrt{3}}+\ln\left|2\sqrt{3}-3\right|.\end{aligned}$$

23. We use integration by parts: $u = x^2$ and $dv = x\cos\left(x^2\right)dx$ so that $du = 2x\,dx$ and $v = \frac{1}{2}\sin\left(x^2\right)$. Then the antiderivative is

$$\int x^3\cos\left(x^2\right)dx = \frac{1}{2}x^2\sin\left(x^2\right) - \int x\sin\left(x^2\right)dx = \frac{1}{2}x^2\sin\left(x^2\right) + \frac{1}{2}\cos\left(x^2\right) + C.$$

The definite integral is therefore

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} x^3\cos\left(x^2\right)dx = \frac{1}{2}\left(x^2\sin\left(x^2\right) + \cos\left(x^2\right)\right)\Big|_{\sqrt{\pi}}^{\sqrt{2\pi}} = \frac{1}{2}(0+1+0+1) = 1.$$

24. The curve $r = 5(\sec\theta + \tan\theta)$ has a removable point of discontinuity at $\theta = 3\pi/2$. Hence, we define the piecewise curve in the problem so that its graph will indeed make a continuous loop. The loop is determined by the angles $\theta = \pi$ and $\theta = 2\pi$. The area of the loop is

$$\begin{aligned}\frac{1}{2}\int_{\pi}^{2\pi} 25(\sec\theta + \tan\theta)^2\,d\theta &= \frac{25}{2}\int_{\pi}^{2\pi}\left(\sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta\right)d\theta \\ &= \frac{25}{2}\int_{\pi}^{2\pi}\left(2\sec^2\theta - 1 + 2\sec\theta\tan\theta\right)d\theta \\ &= \frac{25}{2}\left(2\tan\theta - \theta + 2\sec\theta\right)\Big|_{\pi}^{2\pi} \\ &= \frac{25}{2}(-2\pi + 2 + \pi + 2) = \frac{25}{2}(4 - \pi).\end{aligned}$$

25. Once more, we use integration by parts: $u = \ln x$ and $dv = x^{n-1}$ so that $du = dx/x$ and $v = x^n/n$. Thus, we obtain

$$\int x^{n-1}\ln x\,dx = \frac{x^n}{n}\cdot\ln x - \int\frac{x^{n-1}}{n}\,dx = \frac{x^n\ln x}{n} - \frac{x^n}{n^2} + C.$$

26. The side of the rectangle in the plane of the base has length $\sqrt{y}-y^2$. Thus the height is $420\pi\ln 5\left(\sqrt{y}-y^2\right)$. The volume is

$$\begin{aligned}420\pi\ln 5\int_0^1\left(\sqrt{y}-y^2\right)^2\,dy &= 420\pi\ln 5\int_0^1\left(y-2y^{5/2}+y^4\right)dy\\ &= 420\pi\ln 5\left(\frac{1}{2}y^2-\frac{4}{7}y^{7/2}+\frac{1}{5}y^5\right)\Bigg|_0^1\\ &= 420\pi\ln 5\left(\frac{1}{2}-\frac{4}{7}+\frac{1}{5}\right)\\ &= 420\pi\ln 5\cdot\frac{9}{70}=54\pi\ln 5.\end{aligned}$$

27. We compute the definite integral to obtain an equation in *a*.

$$\begin{aligned}\int_{-a}^{2a}\left(x^3-(a+1)x^2-3ax+4a\right)dx &= \frac{1}{4}x^4-\frac{a+1}{3}x^3-\frac{3a}{2}x^2+4ax\Bigg|_{-a}^{2a}\\ &= 4a^4-\frac{8}{3}\left(a^4+a^3\right)-6a^3+8a^2-\frac{1}{4}a^4-\frac{1}{3}\left(a^4+a^3\right)+\frac{3}{2}a^3+4a^2\\ &= \frac{3}{4}a^4-\frac{15}{2}a^3+12a^2\end{aligned}$$

Setting this equal to zero and clearing fractions gives us $3a^4-30a^3+48a^2=0$, or $a^4-10a^3+16a^2=0$. Since *a* must be a positive number, we know $a\neq 0$, and we may divide by a^2 to get $a^2-10a+16=0$. The solutions are 2 and 8; their sum is 10. (Or, one could use Viete's relations to get 10.)

28. As given, the double integral cannot be evaluated exactly. But we can, by Fubini's Theorem, change the order of integration. The region is $0\le x\le\pi,\ x\le y\le\pi$, which is the triangle in the first quadrant bounded by $y=x,\ y=\pi$, and the *y*-axis. With respect to *x* and then *y*, the region is $0\le y\le\pi,\ 0\le x\le y$. Therefore,

$$\int_0^\pi\int_0^y\frac{\sin y}{y}\,dx\,dy=\int_0^\pi\frac{x\sin y}{y}\Bigg|_0^y\,dy=\int_0^\pi\sin y\,dy=2.$$

29. We use the substitution $u = \sec x \tan x$ so that $du = \left(\sec x \tan^2 x + \sec^3 x\right) dx$. Note that the derivative of $\ln|\sec x \tan x|$ is

$$\frac{d}{dx}\ln|\sec x \tan x| = \frac{\sec x \tan^2 x + \sec^3 x}{\sec x \tan x} = \tan x + \sec^3 x \cot x,$$

so, in terms of u, the indefinite integral is

$$\int\left(\sec x \tan^2 x + \sec^3 x - \tan x - \sec^2 x \cot x\right) dx = \int\left(du + \frac{du}{u}\right) = \int\left(1+\frac{1}{u}\right) du = u + \ln|u| + C.$$

When $x = \pi/6$ then $u = 2/3$, and when $x = \pi/4$ then $u = \sqrt{2}$. Finally, the definite integral is equal to

$$\int_{2/3}^{\sqrt{2}}\left(1+\frac{1}{u}\right) du = u + \ln|u| \Big|_{2/3}^{\sqrt{2}} = \sqrt{2} + \ln\sqrt{2} - \frac{2}{3} - \ln\frac{2}{3} = \sqrt{2} - \frac{2}{3} + \ln\frac{\sqrt{2}}{3}.$$

30. Let $I = \int_0^\infty e^{-x^2}\, dx$ and consider I^2. This can be written as follows, since the variable may be altered to any dummy variable without changing the value of the integral.

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-x^2}\, dx\right) = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = \int_0^\infty\int_0^\infty e^{-x^2-y^2}\, dx\, dy.$$

To evaluate this integral, we convert to polar coordinates. Note that the region in the xy-plane is the first quadrant, and that, in polar, $r^2 = x^2 + y^2$. Thus, the double integral becomes

$$\int_0^{\pi/2}\int_0^\infty re^{-r^2}\, dr\, d\theta,$$

where the factor of r in the integrand is the required Jacobian of the polar change of coordinates. Hence,

$$I^2 = \int_0^{\pi/2}\int_0^\infty re^{-r^2}\, dr\, d\theta = \int_0^{\pi/2}\lim_{b\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^b\, d\theta = \int_0^{\pi/2}\lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) d\theta = \int_0^{\pi/2}\frac{1}{2}\, d\theta = \frac{\pi}{4}.$$

Thus the original integral is $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.