1. C
2. D
3. C
4. D
5. C
6. D
7. B
8. B
9. C
10. D
11. B
12. D
13. C
14. A
15. A
16. D
17. B
18. C
19. B
20. B
21. D
22. E
23. D
24. C
25. B
26. B
27. C
28. A
29. D
30. A

1. C The polynomial factors to $(2x-1)(x+2)(x-2)$, so $x = \frac{1}{2}, -2, 2$, but 2 is the only positive solution.

2. D Over $\mathbb{Z}_7$, the polynomial can be factored to $(2x-1)(x^2-4)$. So $2x \equiv 1$ or $x^2 \equiv 4$ modulo 7. The solutions are $x = 4, 2, 5$.

3. C The question is equivalently asking $19x \equiv 13\ mod\ 29$, which can be rewritten as $19x = 29a + 13$. Taking both sides modulo 19 to get $0 \equiv 10a + 13\ mod\ 19$, or $19b = 10a + 13$. Taking both sides modulo 10 to get $9b \equiv 3\ mod\ 10$. At this point, $b = 7$, substituting back to get $a = 12$ and $x = 19$.

4. D Subtracting 5 from both sides of the original equation, $6x \equiv 18\ mod\ 10$, but since $\gcd(6, 10) = 2$, that yields $x \equiv 3\ mod\ 5$. $23 + 28 + 33 + 38 + 43 + 48 = 213$.

5. C Given that $3x + 5y \equiv 0\ mod\ 37$, we have $3x \equiv -5y\ mod\ 37$, so $40x \equiv -5y\ mod\ 37$, or $y \equiv -8x\ mod\ 37$. Then $ax + 7y \equiv ax - 56y\ mod\ 37$. For that to always be divisible by 37, we have $a \equiv 19\ mod\ 37$.

6. D x must divide into the pair-wise difference of the numbers given. $87937 - 59117 = 28820$, and $131167 - 87937 = 43230$. Therefore, the largest value of x is $\gcd(28820, 43230) = 14410$. The sum of the digits is 10.

7. B Getting common denominator on the left, $\frac{a+b}{ab} = \frac{1}{12}$. Cross multiply and rearrange to get $ab - 12a - 12b = 0$, adding 144 to both sides and factor to get $(a-12)(b-12) = 144$. $144 = 2^4 \cdot 3^2$, so it has 15 factors. $a - 12$ can be equal to each of the 15 factors, with a corresponding value for $b - 12$.

8. B Taking the entire equation modulo 13 to get $5y \equiv 4\ mod\ 13$. By inspection, $y \equiv 6$. This yields two solutions, $(55, 6)$ and $(24, 19)$. $55 + 6 + 24 + 19 = 104$.

9. C Based on the information given, $x = 16A + 4B + C$ and $y = 36A + 6B + C$, so $32A + 8B + 2C = 36A + 6B + C$, or $4A - 2B - C = 0$. Since $4A$ and $2B$ are both even, C must also be even. If $C = 0$, then $B = 2, A = 1$, and $x = 24$. If $C = 2$, there are two possibilities as B can be either 1 or 3. A is equal to 1 or 2, respectively. So the two values for x are 22 and 46. Therefore, $z = 92$, and sum of the digits is 11.

10. D Note that the norm of a Gaussian integer is the square of the more familiar complex norm. The norm of a product is simply the product of the norms. So the norm is $5 \cdot 8 \cdot 50 \cdot 10 = 20000$

11. B A real number is a Gaussian prime if and only if it is a prime that is 3 modulo 4. Otherwise, it is a sum of squares, and can be expressed as a product of two conjugates. For example, $2 = (1+i)(1-i), 5 = (2+i)(2-i)$. (Similarly for purely imaginary numbers.
A Gaussian integer with both real and imaginary parts is prime if its norm is a prime number (call it p). Then if it is expressed as a product of two Gaussian integers, their norms must be 1 and p. The norm of $3+5i$ is 34, so it can be expressed as the product of two Gaussian integers with norms 2 and 17. One such expression is $(1+i)(4+i)$. $2+3i, 2+5i$, and 3 are the Gaussian primes in the set.

12. D All but the sets of real numbers and the set of irrational numbers are countably infinite.

13. C $720 = 2^4 \cdot 3^2 \cdot 5$, so a and b must only have 2, 3, and 5 in its prime factorization. Consider powers of 2 for a and b. One of them must have 2^4 in its prime factorization, or their lcm cannot have 2^4. The other can have any power of 2 from 0 to 4. So the number of ways for a and b to have powers of 2 is $2 \cdot 5 - 1 = 9$. The case where both have 2^4 is double counted, so it must be subtracted out. Similarly, there are $2 \cdot 3 - 1 = 5$ ways for powers of 3, and $2 \cdot 2 - 1 = 3$ ways for powers of 5. All 3 are independent of each other, making a total of $9 \cdot 5 \cdot 3$ ordered pairs.

14. A Based on the information given,
$$\frac{b^2+3}{b^2+3b+6} = \frac{b+4}{b+8}$$
Cross multiply to get $b^3 + 8b^2 + 3b + 24 = b^3 + 7b^2 + 18b + 24$, or $b^2 - 15b = 0$. Therefore, $149_{15} = 225 + 60 + 9 = 294$ and $338_{15} = 675 + 45 + 8 = 728$. $\frac{294}{728} = \frac{21}{52} = \left(\frac{16}{37}\right)_{15}$.

15. A It's fastest to consider this problem in chunks of 30. For every 30 consecutive integers, there are 15 multiples of 2, 10 multiples of 3, 5 multiples of 6, 3 multiples of 10, 2 multiples of 15, and 1 multiple of 30. So the number of integers satisfying the conditions given is $15 + 10 - 5 - (3 + 2 - 1) = 16$. From 7 to 2016, there are $67 \cdot 16 = 1072$ such numbers, then tack on 2, 3, 4, and 6 for 1076 such numbers.

16. D Euler's totient theorem does not apply directly, since $\gcd(8, 800) \neq 1$. So we must consider 8^{321} modulo 32 and 25, since $800 = 2^5 \cdot 5^2$. Clearly, $8^{321} \equiv 0 \; mod \; 32$. $\phi(25) = 20$, so $8^{20} \equiv 1 \; mod \; 25$, or $8^{321} \equiv (8^{20})^{16} \cdot 8 \equiv 8 \; mod \; 25$. Now we are seeking a number less than 800 that is 0 mod 32 and 8 mod 25, and that number is 608.

17. B Let $x = p^a q^b$, where p, q are distinct primes. Then $N = (a+1)(b+1)$.
$x^2 = p^{2a} q^{2b}$, so $3N = (2a+1)(2b+1)$.
Therefore, $3ab + 3a + 3b + 3 = 4ab + 2a + 2b + 1$,
Or $ab - a - b = 2$. Adding 1 to both sides, $(a-1)(b-1) = 3$.
Since both a and b are positive integers (or x wouldn't have two prime factors), one of them is 2, the other is 4, and $N = 15$.
$x^7 = p^{14} q^{28}$, which has $(14+1)(28+1) = 29N$ factors.

18. C For $x^k \equiv k \ mod \ 363$ to be true in general, $x^2 \equiv x \ mod \ 363$. So $x^2 - x = x(x-1)$ must be divisible by 363. Since $363 = 3 \cdot 11^2$, either one of x and $x - 1$ is divisible by 363, or one of them is divisible by 121 and the other is divisible by 3. This leaves 4 possibilities modulo 363, which are $x \equiv 0, 1, 121, 243 \ mod \ 363$. There are 3 of each of the 4 possibilities, since $243 + 363 \cdot 2 < 1000$. Their sum is $3(0 + 1 + 121 + 243) + 4(363 + 2 \cdot 363) = 5451$. $5^2 + 4^2 + 5^2 + 1^2 = 67$.

19. B By Wilson's theorem, $(p-1)! \equiv -1 \ mod \ p$. So $r(f(n), n)$ is $n - 1$ if n is prime. If n is not prime, $r(f(n), n) = 0$, since n is part of $(n+1)!$. So $\sum_{n=1}^{25} r(f(n), n) = 91$. You may remember that sum of primes under 25 is 100, and there are 9 of them. Alternatively, just add them up. 91 leaves a remainder of 1 when divided by 5.

20. B

$$\frac{9}{34} = \frac{1}{\frac{34}{9}} = \frac{1}{3 + \frac{7}{9}} = \frac{1}{3 + \frac{1}{\frac{9}{7}}} = \frac{1}{3 + \frac{1}{1 + \frac{2}{7}}} = \frac{1}{3 + \frac{1}{1 + \frac{1}{\frac{7}{2}}}} = \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \frac{1}{2}}}}$$

So $\frac{9}{34} = [0; 3, 1, 3, 2]$. $3^2 + 1^2 + 3^2 + 2^2 = 23$.

21. D Let $x = [2; \overline{1, 2}]$. Then

$$x = 2 + \frac{1}{1 + \frac{1}{\mathbf{2} + \frac{\mathbf{1}}{\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2} + \cdots}}}}$$

Substituting x for the bolded part, we have

$$x = 2 + \frac{1}{1 + \frac{1}{x}} = 2 + \frac{x}{x+1} = \frac{3x + 2}{x + 1}$$

So $x^2 + x = 3x + 2$, or $x^2 - 2x = 2$.
Completing the square to get $(x-1)^2 = 3$, and since $2 < x < 3$, $x = 1 + \sqrt{3}$.

22. E Convergents are the truncated continued fractions. Note that $3 = 3$, $\frac{22}{7} = 3 + \frac{1}{7}$, and $\frac{355}{113} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}}$. So the third convergent is $3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106}$.

23. D When the convergents are written as $\frac{x}{y}$, the solutions to $x^2 - 2y^2 = \pm 1$, alternating between 1 and -1, starting with $1^2 - 2 \cdot 1^2 = -1$ The first six convergents of $\sqrt{2}$ are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, and $\frac{99}{70}$. $(99, 70)$ is the third smallest solution to $x^2 - 2y^2 = 1$.

24. C Each element r of the list is in the form of $\frac{p}{q}$, where p is a factor of $5400 = 2^3 3^3 5^2$, and q is a factor of $2880 = 2^6 3^2 5$. So $r = \pm 2^a 3^b 5^c$, where $-6 \leq a \leq 3, -2 \leq b \leq 3$, and $-1 \leq c \leq 2$, so the total number of possibilities is $2 \cdot 10 \cdot 6 \cdot 4$, and with 6 roots, the probability of picking a root is 1/80.

25. B Dropping height from A to CD, and call the foot F, possibly coinciding with D. Since $m\angle C < m\angle D$, $DF < CE$. Call, $DF = x, AD = a, BC = b$, and height h, not necessarily rational. Then $CE = 6 - x$, and x can only be 0, 1, or 2. Applying Pythagorean Theorem to ΔAFD and ΔBEC, $x^2 + h^2 = a^2, (6-x)^2 + h^2 = b^2$. Subtracting the first from the second to arrive at $36 - 12x = (b-a)(b+a)$.
Now consider the cases:
1. $x = 0$, then $(b-a)(b+a) = 36$, for both a and b to be positive integers, $b - a = 2$ and $b + a = 18$, or $b = 10, a = 8$.
2. $x = 1$, then $(b-a)(b+a) = 24$, there are 2 subcases:
    a. $b - a = 2, b + a = 12$, then $b = 7, a = 5$
    b. $b - a = 4, b + a = 6$, then $b = 5, a = 1$, but this makes the trapezoid degenerate, and thus is not a solution.
3. $x = 2$, then $(b-a)(b+a) = 12$, then $b - a = 2, b + a = 6$, or $b = 4, a = 2$. This is also degenerate.

This leaves a total of 2 possible trapezoids.

26. B Let $x^3 = 31{,}217{,}193{,}218{,}303$. Clearly, the unit digit of x must be 7. Eliminate E.
Consider x^3 modulo 8. It is 7 mod 8, so x must be 7 mod 8. Eliminate A.
x^3 is 8 mod 9, so x must be 2 mod 3. Eliminate C.
Finally, $32087^3 > 32000^3 = 1024 \cdot 32 \cdot 10^9 > 32 \cdot 10^{12}$. Eliminate D.

27. C $30 = 2 \cdot 3 \cdot 5$, and $54 = 2 \cdot 3^3$. For $30|54N$, N must have a factor of 5. For $54|30N$, N must have factors of 3^2. Therefore, N must be a multiple of 45. Further, N divides $54 \cdot 30 = 1620$, so N must also be a factor of 1620. $1620 = 2^2 \cdot 3^4 \cdot 5$, reserving $3^2 \cdot 5$ to ensure a multiple of 45 leaves $2^2 \cdot 3^2$, so there are $(2+1)(2+1) = 9$ possibilities for N.

28. A When substituting $a \cdot b^n + c \cdot d^n$ into the recurrence, the powers of b and d stay separate, and a and c factor out, so simply substituting in b^n is sufficient for solving for b and d. $G_n = 3G_{n-1} + 4G_{n-2}$ becomes $b^n = 3b^{n-1} + 4b^{n-2}$, or $b^{n-2}(b^2 - 3b - 4) = 0$. This yields two real solutions, which are the values of b and d, and they add to 3.
Taking the recurrence back one step (using $n = 2$), we have $13 = 3 \cdot 7 + 4G_0$, or $G_0 = -2$. Substituting in 0 for n in the explicit formula to get $a + c = -2$.
So $a + b + c + d = 1$.

29. D $0.1_{10} = \left(0.0\overline{0011}\right)_2$, so $m = 1, n = 4$, and $\sum_{k=1}^{n} A_{m+k} 2^k = 8 + 16 = 24$.

30. A $A = 4, B = 4, C = 2, D = 0$, the sum is 10.