Answers:

1. D
2. B
3. C
4. A
5. B
6. A
7. D
8. B
9. C
10. A
11. C
12. D
13. B
14. C
15. D
16. D
17. B
18. B
19. C
20. A
21. C
22. D
23. A
24. B
25. D
26. A
27. D
28. C
29. A
30. B

Solutions:

1. There are two options (heads or tails) for each flip, so there are a total of $2^4 = 16$ different outcomes.

2. If the first card is the Ace of Spades, then there are three aces left in the remaining 51 cards, so the probability is $3/51 = 1/17$.

3. The probability of rolling a 5 or 6 is $2/6 = 1/3$, so you should expect to roll a 5 or 6 in 3 rolls.

4. Since $600 = 2^3 \cdot 3 \cdot 5^2$, the total number of positive integral divisors of 600 is $4 \cdot 2 \cdot 3 = 24$. Therefore, the probability is $\frac{24 \cdot 1}{24 \cdot 23} = \frac{1}{23}$.

5. The probability distribution table for this experiment is:

X	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
p	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

So multiplying each value of *X* by its corresponding value of *p* and summing these, the expected value is

$$\frac{1+4+6+12+10+24+16+9+20+48+30+16+36+40+48+25+60+36}{36} = \frac{441}{36} = 12.25.$$

6. The probability distribution table for this experiment is:

X	0	1	2	3	4	5
p	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

So multiplying each value of *X* by its corresponding value of *p* and summing these, the expected value is $\frac{0+10+16+18+16+10}{36} = \frac{70}{36} = \frac{35}{18}$.

7. The probability distribution table for this experiment is:

X	1	2	3	4	5	6
p	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

So multiplying each value of X by its corresponding value of p and summing these, the expected value is $\frac{1+6+15+28+45+66}{36}=\frac{161}{36}$.

8. Let k be the number of boys, where $k<n$. Therefore, $\frac{\binom{n}{k}}{2^n}=\frac{\binom{n+1}{k+1}}{2^{n+1}}$

$\Rightarrow\frac{2\cdot n!}{k!(n-k)!}=\frac{(n+1)!}{(k+1)!(n-k)!}\Rightarrow\frac{2}{n+1}=\frac{1}{k+1}\Rightarrow n=2k+1$. Since k was the number of boys originally, $k+1$ is the original number of girls, meaning there was 1 extra girl initially.

9. If a head never comes immediately after a tail, then once a tail is flipped, every flip afterward is a tail. Therefore, the sequences that are possible begin with anywhere from 0 to 5 heads and finish out with tails. Therefore, the probability is $6/2^5=3/16$.

10. The die must come up as 2, 4, or 6 to be able to have an equal numbers of heads as tails.

Therefore, the probability is $\frac{1}{6}\cdot\left(\frac{\binom{2}{1}}{2^2}+\frac{\binom{4}{2}}{2^4}+\frac{\binom{6}{3}}{2^6}\right)=\frac{2\cdot16+6\cdot4+20}{6\cdot64}=\frac{76}{384}=\frac{19}{96}$.

11. Let x be the number of people who like both Transformers and Gobots. Since liking each type of toy are independent events, $\frac{42+x}{100}\cdot\frac{12+x}{100}=\frac{x}{100}\Rightarrow 0=x^2-46x+504=(x-28)(x-18)$ $\Rightarrow x=18$ or $x=28$. The greater number is 28.

12. Case 1 (a blue marble is transferred from Vase A to Vase B): $p=\frac{3}{5}\cdot\frac{6}{7}=\frac{18}{35}$

Case 2 (a green marble is transferred from Vase A to Vase B): $p=\frac{2}{5}\cdot\frac{5}{7}=\frac{10}{35}$

Case 3 (a blue marble is transferred from Vase B to Vase A): $p=\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$

Case 4 (a green marble is transferred from Vase B to Vase A): $p=\frac{1}{6}\cdot\frac{3}{6}=\frac{3}{36}$

Therefore, the probability that the transferred marble was green, transferred from Vase A to Vase B, is $\frac{10/35}{18/35+10/35+20/36+3/36}=\frac{2/7}{259/180}=\frac{2}{7}\cdot\frac{180}{259}=\frac{360}{1813}$.

13. In order for the real number to be less than $1/3$ away from an integer, the number must be in the intervals $\left[0,1/3\right)\cup\left(2/3,1\right]$. The sum of the lengths of the intervals is $2/3$, and the length of the interval from which the real number can come has length 1, so the probability is $2/3$.

14. The probability is $0.55+0.62-0.48\cdot 0.62=0.8724$

15. The probability is $0.2\cdot 0.4+0.25\cdot 0.3+0.1\cdot 0.3+0.45\cdot 0.5=0.08+0.075+0.03+0.225=0.41$.

16. $p\left(X>2000\,|\,X>1500\right)=\dfrac{p\left(X>2000\cap X>1500\right)}{p\left(X>1500\right)}=\dfrac{p\left(X>2000\right)}{p\left(X>1500\right)}=\dfrac{0.15}{0.2}=0.75$

17. If you win on the first try, you must have picked a goat first, then Monty Hall must have shown the other goat, so the probability is $\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}$. If you win on the second try, you must have picked a goat first, then Monty Hall must have shown the car, followed by you picking a goat first, then Monty Hall shows the other goat, so the probability is $\left(\frac{2}{3}\cdot\frac{1}{2}\right)^2=\frac{1}{9}$. Continuing in this way, to win on the nth try, the probability is $\frac{1}{3^n}$, so the probability of winning is

$$\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1/3}{1-1/3}=\frac{1/3}{2/3}=\frac{1}{2}.$$

18. Let $S=\sum_{n=1}^{\infty}\left(\left(2n+3\right)\left(a/b\right)^n\right)$. Then $\frac{a}{b}\cdot S=\sum_{n=1}^{\infty}\left(\left(2n+3\right)\left(a/b\right)^{n+1}\right)$ and $S\left(1-\frac{a}{b}\right)=S-\frac{a}{b}\cdot S$

$$=5\cdot\frac{a}{b}+2\left(\frac{a}{b}\right)^2+2\left(\frac{a}{b}\right)^3+\ldots=\frac{5a}{b}+\frac{2a^2}{b^2\left(1-\frac{a}{b}\right)}=\frac{5a}{b}+\frac{2a^2}{b\left(b-a\right)}=\frac{a\left(5b-3a\right)}{b\left(b-a\right)}\Rightarrow S=\frac{a\left(5b-3a\right)}{\left(b-a\right)^2}.$$

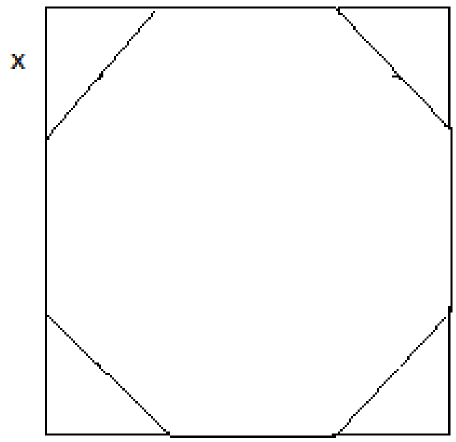
Considering the 9 possible combinations of values for a and b for which $\frac{a}{b}$ is reduced, there are 6 possible combinations that make this quantity an integer ($(1,2)$, $(1,3)$, $(2,3)$, $(3,4)$, $(3,5)$, and $(4,5)$), so the probability is $\frac{6}{9}=\frac{2}{3}$.

19. The dartboard looks like the picture to the right, and let x be the length of a leg of the corner right triangle. Then the total area of the board is

$\left(2x+x\sqrt{2}\right)^2=x^2\left(6+4\sqrt{2}\right)$. The area inside the octagon is $x^2\left(6+4\sqrt{2}\right)-2x^2=x^2\left(4+4\sqrt{2}\right)$. The probability of a throw landing inside the octagon is $\frac{4+4\sqrt{2}}{6+4\sqrt{2}}=2\left(\sqrt{2}-1\right)$. The probability of a throw landing outside the octagon is $\frac{2}{6+4\sqrt{2}}=3-2\sqrt{2}$.

Therefore, the probability of exactly two of Gary's darts landing inside the octagon is

$3\left(3-2\sqrt{2}\right)\left(2\sqrt{2}-2\right)^2=204-144\sqrt{2}$.



20. This is the number of derangements of 6 cards, which is 265. Therefore, the probability is $\frac{265}{6!}=\frac{53}{144}$.

21. The measure of an exterior angle is $\frac{360^\circ}{n}$, where $n\geq 3$. Since $360=2^3\cdot 3^2\cdot 5$, there are a total of $4\cdot 3\cdot 2=24$ positive integral factors of 360. Since we must exclude 1 and 2, there are a total of 22 possible values of *n*.

22. Let *n* be the number of other-colored balls. There are six different orders for drawing one red, one black, and one other-colored ball, and the probabilities of these orders are as follows:

RBO: $\frac{4}{n+8}\cdot\frac{5}{n+9}\cdot\frac{n}{n+9}=\frac{20n}{(n+8)(n+9)^2}$ BRO: $\frac{4}{n+8}\cdot\frac{5}{n+8}\cdot\frac{n}{n+9}=\frac{20n}{(n+8)^2(n+9)}$

ROB: $\frac{4}{n+8}\cdot\frac{n}{n+9}\cdot\frac{5}{n+8}=\frac{20n}{(n+8)^2(n+9)}$ BOR: $\frac{4}{n+8}\cdot\frac{n}{n+8}\cdot\frac{5}{n+7}=\frac{20n}{(n+7)(n+8)^2}$

ORB: $\frac{n}{n+8}\cdot\frac{4}{n+7}\cdot\frac{5}{n+8}=\frac{20n}{(n+7)(n+8)^2}$ OBR: $\frac{n}{n+8}\cdot\frac{4}{n+7}\cdot\frac{5}{n+7}=\frac{20n}{(n+7)^2(n+8)}$

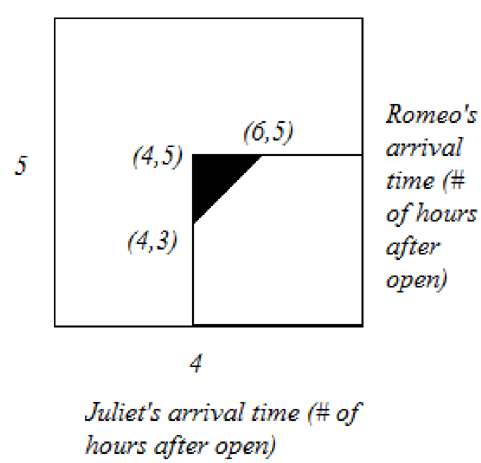
The probability is the sum of these, which after factoring and using partial fraction decomposition becomes:

$$\frac{20n}{n+8}\left(\frac{1}{(n+9)^2}+\frac{2}{(n+8)(n+9)}+\frac{2}{(n+7)(n+8)}+\frac{1}{(n+7)^2}\right)=\frac{20n}{n+8}\left(\frac{1}{(n+9)^2}+\frac{2}{n+8}-\frac{2}{n+9}\right.$$

$\left. +\frac{2}{n+7}-\frac{2}{n+8}+\frac{1}{(n+7)^2}\right)=\frac{20n}{n+8}\left(\frac{1}{(n+9)^2}-\frac{2}{n+9}+\frac{2}{n+7}+\frac{1}{(n+7)^2}\right)$. By inspection, this equals $\frac{3068}{13005}$ when $n=8$ (additionally, this could be guessed by observing that $13005=5\cdot 3^2\cdot 17^2$, so the only factoring that would feature three consecutive integers, with possibly some cancelling, is $15^2\cdot 16\cdot 17^2$, which would make $n+7=15,\ n+8=16,$ and $n+9=17\Rightarrow n=8$).

23. Moe wins if he rolls a 2, 3, or 5, so he has probability of winning $3/6=1/2$. Larry wins if he rolls a 4 or 6, so he has probability of winning $2/6=1/3$. Curly wins if he rolls a 1, so he has probability of winning $1/6$. The probability that Moe will win is $\frac{1}{2}+\left(\frac{1}{2}\right)^2\cdot\frac{2}{3}\cdot\frac{5}{6}+\left(\frac{1}{2}\right)^3\cdot\left(\frac{2}{3}\cdot\frac{5}{6}\right)^2+\ldots=\frac{\frac{1}{2}}{1-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{5}{6}}=\frac{9}{13}$. The probability that Curly will win is $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{6}+\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2\cdot\frac{5}{6}\cdot\frac{1}{6}+\left(\frac{1}{2}\cdot\frac{2}{3}\right)^3\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+\ldots=\frac{\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{6}}{1-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{5}{6}}=\frac{1}{13}$. Therefore, Moe is 9 times as likely to win as Curly.

24. First, there are a total of $55\cdot 8=440$ instances of symbols. Suppose only 54 different symbols were necessary; in this instance, the symbols would show up an average of $\frac{440}{54}=8.14\ldots$ times—in other words, some symbols show up at least nine times. Find the nine cards that share a common symbol. All of the other symbols on those cards must all be different, which means a total of $9\cdot 7+1=64$ different symbols must be necessary, a contradiction. If we up this to 56 symbols, by similar argument, a total of $8\cdot 7+1=57$ symbols would be necessary, still a contradiction. If we suppose 57 symbols were necessary, a total of $8\cdot 7+1=57$ symbols would be necessary, which is not a contradiction. Therefore, the minimum number of symbols needed is 57.



25. In the diagram, the *x*-axis represents Juliet's arrival time in the number of hours after the store opens,

while the y-axis represents Romeo's arrival time in the number of hours after the store opens. The lower right 5 by 5 square represents the possible combinations of arrival times, while the shaded right triangle represents the times when Romeo and Juliet will be at the store at the same time. Therefore, the probability that Romeo and Juliet are at the store at the same time for some amount of time is $\frac{\frac{1}{2}\cdot 2\cdot 2}{5\cdot 5}=\frac{2}{25}$.

26. In each box, there is a $49/50$ probability that the fake coin will not be found. Therefore, the probability that the whole ruse goes undetected is $\left(\frac{49}{50}\right)^{50}=\left(1+\frac{-1}{50}\right)^{50}$, which is incredibly close to $e^{-1}=\frac{1}{e}$ (recall that $e^{ab}=\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$).

27. Assign to each child an ordered pair where the first entry indicates sex and the second entry indicates season of birth. Therefore, with the given information, the possible orderings of Mrs. Brady's children are as follows:

Case 1: (F,F), (F,Su), ____ (8 possibilities)
Case 2: (F,Su), (F,F), ____ (8 possibilities)
Case 3: (F,F), ____, (F,Su) (8 possibilities)
Case 4: (F,Su), ____, (F,F) (8 possibilities)
Case 5: ____, (F,Su), (F,F) (8 possibilities)
Case 6: ____, (F,F), (F,Su) (8 possibilities)

However, some of these cases overlap sometimes. Each of these overlap once: Cases 1 & 3, Cases 1 & 5, Cases 2 & 4, Cases 2 & 6, Cases 3 & 6, and Cases 4 & 5. Therefore, there are a total of $48-6=42$ cases. Half of the full number of cases have three girls, but the overlaps are each double-counted too, so the number of cases with three girls is $24-6=18$. The probability, therefore, is $18/42=3/7$.

28. Since all probabilities will have denominator $\binom{52}{7}=133{,}784{,}560$, we need just examine how many ways to form all of the different hands. The number of ways of being dealt an Uncle Jesse is $\binom{13}{4}\binom{4}{3}\binom{4}{2}^3\binom{4}{1}=2{,}471{,}040$. For 5-card hands, a full house can be dealt in three ways: (1) a three-of-a-kind, a pair, and two kickers; (2) one three-of-a-kind and two pairs (note that this is NOT an Uncle Jesse); or (3) two threes-of-a-kind and one kicker. For case (1), the total number of ways is $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{2}\binom{4}{1}^2=3{,}294{,}720$; for case (2), the total number

of ways is $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2 = 123,552$; for case (3), the total number of ways is $\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{1} = 54,912$; therefore, the total number of ways of being dealt a full house is $3,294,720 + 123,552 + 54,912 = 3,473,184$, meaning the Uncle Jesse is a better hand than the full house. Also, for 5-card hands, the number of ways of being dealt a four-of-a-kind is $\binom{13}{1}\binom{4}{4}\binom{48}{3} = 224,848$, so the four-of-a-kind beats the Uncle Jesse. Since the full house and the four-of-a-kind are next to each other in the order of hands (just as with 5-card poker), the least hand that beats the Uncle Jesse is the four-of-a-kind.

29. The list of possible outcomes where the results sum exactly to 14 is:

Case 1: 1,1,6,6 (6 sequences)	Case 7: 2,2,5,5 (6 sequences)
Case 2: 1,2,5,6 (24 sequences)	Case 8: 2,3,3,6 (12 sequences)
Case 3: 1,3,4,6 (24 sequences)	Case 9: 2,3,4,5 (24 sequences)
Case 4: 1,3,5,5 (12 sequences)	Case 10: 2,4,4,4 (4 sequences)
Case 5: 1,4,4,5 (12 sequences)	Case 11: 3,3,3,5 (4 sequences)
Case 6: 2,2,4,6 (12 sequences)	Case 12: 3,3,4,4 (6 sequences)

Therefore, the probability of rolling a sequence where the sum is exactly 14 is:

$$\frac{3\cdot 6 + 3\cdot 24 + 4\cdot 12 + 2\cdot 4}{6^4} = \frac{73}{648}.$$

30. This is actually a semi-famous problem; the non-standard dice are called “Sicherman dice”. Using the argument of generating functions, for standard dice, let $x + x^2 + x^3 + x^4 + x^5 + x^6$ be the generating function, where the exponent represents the face rolled and each coefficient is the frequency that that face occurs. Multiplying this function by itself yields $x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$, so the exponents of this polynomial represent the sum while the coefficients represent the corresponding frequencies. We need to find another factorization of this polynomial. $\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2 = x^2\left(1 + x + x^2\right)^2\left(1 + x\right)^2 \left(1 - x + x^2\right)^2$, so we need to just find a new way to combine these factors into two polynomials. The two factors of x must be separated in order for there to be no constant in either polynomial. Additionally, each polynomial must be such that the coefficients sum to 6 (since there are six faces on each die). In the same order as above, the factors, when 1 is plugged in, evaluate to 1, 3, 2, and 1. Therefore the two polynomials are:

$x\left(1+x+x^2\right)\left(1+x\right)$________ and $x\left(1+x+x^2\right)\left(1+x\right)$________ , with the remaining factors left to place. If one was placed in each blank, then the two dice would be standard dice as that was the original factoring. Therefore, the two polynomials are $x\left(1+x+x^2\right)\left(1+x\right)=x+2x^2+2x^3+x^4$ and $x\left(1+x+x^2\right)\left(1+x\right)\left(1-x+x^2\right)^2=x+x^3+x^4+x^5+x^6+x^8$, meaning that one die has faces 1, 2, 2, 3, 3, 4 while the other has faces 1, 3, 4, 5, 6, 8. The largest possible product, then, is $4\cdot 8=32$.