The acronym NOTA denotes that None of These Answers is correct. All numbers are assumed real unless otherwise explicitly stated. Good luck and, most importantly, have fun!

Questions 1 and 2 are based on the following scenario.

When employees of Congressional Efficiency, Inc. receive new computers, they are not allowed to choose which software programs they want installed. Rather, they are algorithmically assigned one antivirus package, one statistical computing program, and one web browser. There are 3 types of antivirus package (A, B, or C); 3 types of statistical computing programs (M, R, or S); and 3 types of web browsers (E, F, or G). However, there is nothing to prevent a second of each category from being installed on the computer as well. The following requirements must hold for any given employee's computer:

- The same number of programs from each category must be installed onto the computer
- R and E are made by rival developers and therefore cannot be installed together on the same computer.
- If S is installed on a computer, then F must be installed as well.
- If A is installed on a computer, then M must be installed as well.
- When R is installed onto a computer, S cannot be installed onto the same computer.

1. If 2 browsers are installed on a given computer, then which of the following cannot be true?
A. E and F installed B. M and S installed C. M and R installed D. A and B installed E. NOTA

2. Suppose that R is the only statistical computing program loaded onto an employee's computer. How many possible combinations of programs could be loaded onto the computer?
A. 3 B. 4 C. 5 D. 6 E. NOTA

3. Consider the following "proof" that 1 = 2:

- Step 1: $-\frac{1}{1}=\frac{1}{-1}$
- Step 2: $\sqrt{\frac{-1}{1}}=\sqrt{\frac{1}{-1}}$
- Step 3: $\frac{\sqrt{-1}}{\sqrt{1}}=\frac{\sqrt{1}}{\sqrt{-1}}$
- Step 4: This implies that $\frac{i}{1}=\frac{1}{i}$.
- Step 5: Dividing by 2 and adding $\frac{3}{2i}$ we have $\frac{i}{2}+\frac{3}{2i}=\frac{1}{2i}+\frac{3}{2i}$
- Step 6: Multiplying through by i we have $\frac{i^2}{2}+\frac{3i}{2i}=\frac{i}{2i}+\frac{3i}{2i}$
- Step 7: Simplifying, we have $-\frac{1}{2}+\frac{3}{2}=\frac{1}{2}+\frac{3}{2}\Rightarrow 1=2!$

Where is the flaw?
A. Step 3 B. Step 4 C. Step 5 D. Step 7 E. NOTA

4. You have a standard, unbiased, six-sided die with the integers 1 through 6 on the six sides. You play a game where you receive money equal to the sum of the numbers you roll. However, if the cumulative sum of the numbers you have rolled equals a perfect square, the game ends and you receive zero (for example, if the sum of your previous rolls has been 21 and you roll a 4, the game ends and you get nothing). After each turn, you can also choose to quit and keep what you have rather than to keep rolling. At which of the following numbers would it **NOT** be profitable to continue rolling?
A. 15 B. 24 C. 35 D. 48 E. NOTA

5. Consider the following proof, which may or may not contain a flaw, of the following:

$$\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots}_{n\ terms}=\frac{3}{2}-\frac{1}{n}$$

We can prove this by induction:

- Step 1: For the basis case, n = 1, we have $\frac{1}{1\cdot 2}=\frac{3}{2}-\frac{1}{1}$
- Step 2: Next, in the inductive step, we have

$$\left(\frac{1}{1\cdot 2}+\cdots+\frac{1}{(n-1)\cdot n}\right)+\frac{1}{n\cdot(n+1)}=\frac{3}{2}-\frac{1}{n}+\frac{1}{n\cdot(n+1)}=\frac{3}{2}-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}=\frac{3}{2}-\frac{1}{n+1}$$

Is this proof valid, or if not, what's wrong?

A. Proof is valid B. Step 1 algebra C. Step 2 algebra D. Step 2 indexing E. NOTA

6. The well-ordering principle states that every nonempty subset of the natural numbers has a least element. This seemingly innocuous statement is logically equivalent to which standard proof technique?

A. Contradiction B. Construction C. Induction D. Contrapositive E. NOTA

For question 7, it is assumed that all vectors are of the same dimension.

7. In vector notation, the Cauchy-Schwarz inequality states that for two vectors $\vec{x}$ and $\vec{y}$, we have

$$|\vec{x}\cdot\vec{y}|<\|\vec{x}\|\cdot\|\vec{y}\|$$

where $\vec{x}\cdot\vec{y}$ denotes the dot product of the two vectors, and $\|\vec{z}\|$ denotes the magnitude of a vector $\vec{z}$. Given this, which of the following statements can be proved using the Cauchy-Schwarz inequality?

I. If $a_1+a_2+\cdots+a_n=1$ then $a_1^2+a_2^2+\cdots+a_n^2\geq\frac{1}{n}$ for real numbers $a_1,a_2,\ldots a_n$

II. $(a^2b^2+b^2c^2+c^2d^2+d^2a^2)(ab^3+bc^3+cd^3+da^3)\leq 16a^2b^2c^2$ for real numbers a,b,c

III. The triangle inequality: $\|\vec{x}+\vec{y}\|<\|\vec{x}\|+\|\vec{y}\|$ for any two vectors $\vec{x}$ and $\vec{y}$

A. I and II only B. II and III only C. I and III only D. I, II, III E. NOTA

Use the following for Questions 8 and 9: There are two types of CEOs in the world: workers and shirkers. Three-fourths of CEOs are shirkers; the remaining are workers. When a CEO gives a press conference, however, he or she may make statements that help give away his or her type. In particular, workers will use the word "enhancement" at least once in 20% of press conferences they give, while shirkers will use the word "enhancement" at least once in 60% of press conferences they give. Assume that a CEO's actions are independent across press conferences, i.e. the likelihood of a CEO using the word "enhancement" at a given press conference is independent of whether or not that CEO used the word "enhancement" at any other press conference.

8. You attend two press conferences given by the same CEO and hear the word "enhancement" at least once at both. What is the conditional probability that the CEO is a **shirker**?

A. $\frac{27}{31}$ B. $\frac{8}{9}$ C. $\frac{9}{10}$ D. $\frac{27}{28}$ E. NOTA

9. You attend 2*N* press conferences given by the same CEO and hear the word "enhancement" at least once at *N* of these press conferences. After doing this, you calculate the conditional probability that the CEO is a shirker to be greater than 95%. What is the smallest possible positive integer *N*?

A. 3 B. 4 C. 5 D. 6 E. NOTA

10. A *derangement* of a sequence is a permutation of that sequence where no element remains in its original position. For example, given the sequence ABCDE, BCDEA is a derangement but neither ACBED nor BCEDA are derangements. How many derangements are there of the word WASHING?

A. 1638 B. 1854 C. 4166 D. 4421 E. NOTA

11. There are 75 people who live on an island who convene once a day. Each person is wearing a hat that is either red or blue. These people are bound to silence, and there are no mirrors present; hence, when the 75 people convene on a given day, they can each observe the colors of everyone else's hats except their own. As soon as a person realizes he/she is wearing a red hat, he/she must immediately leave the island and never come back. Before the first day, everyone knows that at least 7 people are wearing red hats. If a total of 22 people are wearing red hats, how many days will it take for everyone wearing a red hat to leave the island? (Assume that as soon as someone discovers that they are wearing a red hat, they immediately leave)

A. 15 B. 16 C. 22 D. 23 E. NOTA

12. You have 50 red marbles and 50 blue marbles. You are required to divide these into two bowls, where each bowl must contain at least one marble. A bowl is then chosen at random, from which a marble is then chosen at random. If this marble is blue, you win \$2; if it is red, you lose \$1. Over the set of all possible strategies, let X be your highest possible expected payoff and let Y be your lowest possible expected payoff. What is $\frac{X}{Y}$?

A. 1 B. $-\frac{98}{17}$ C. $-\frac{41}{8}$ D. $-\frac{25}{4}$ E. NOTA

Use the following for Questions 13 and 14:

There are four managers. Each runs a single different company fund; each of these companies was founded in distinct cities, and each firm made a different amount of profit last year. The four managers are Alice, Bob, Cat, and Prometheus; the four cities are New York, Miami, St. Louis, and San Francisco; the profits are \$1, \$2, \$3, and \$4; and the four companies' names are Average Returns, Magenta Rock, Silverman Rucks, and Capslock Capital. Your job is to match the managers to their company, salary, and founding city, given the following clues:

- The company run by Alice had profits \$2 larger than the one founded in St. Louis.
- Neither the one with profits of \$2 nor the one with profits of \$4 is Magenta Rock.
- Of the one with a return of \$4 and the one founded in New York, one is run by Cat and the other is Average Returns.
- The one founded in Miami is either the one run by Prometheus or Magenta Rock (but not both).
- The one founded in St. Louis had profits \$2 smaller return than Capslock Capital.

13. If the company run by Prometheus earned profits of K, let $K = \frac{M}{N}$ where the fraction is in reduced form as a ratio of positive integers. Let L be the number of consonants in the name of the city that Silverman Rucks was founded in (use "ST. LOUIS" rather than "SAINT LOUIS"). Find $M + N + L$.

A. 8 B. 10 C. 11 D. 12 E. NOTA

14. Capslock Capital earned profits of Q; Prometheus earned profits P. Find $\frac{Q}{P}$.

A. 0.5 B. 0.75 C. 1.5 D. 2 E. NOTA

15. Consider the following game. Ms. Herron and Dr. Morris take turns drawing integers from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, without replacement (so that whoever goes first draws up to five integers while whoever goes second can only draw up to four). One wins by choosing three numbers whose sum is 0. Ms. Herron goes first. If both players play optimally, can she win, and if so, what number should she choose first?

A. -4 B. 0 C. 1 D. Cannot win E. NOTA