Answers:

1. A
2. C
3. D
4. A
5. B
6. B
7. D
8. B
9. E
10. B
11. D
12. C
13. A
14. C
15. B
16. E
17. D
18. D
19. D
20. C
21. A
22. C
23. B
24. A
25. C
26. D
27. B
28. C
29. A
30. E

Solutions:

1. Any number in both sets must be one less than both a multiple of 3 and a multiple of 4; i.e., a multiple of 12. They must also be less than or equal to 50 because of the rule of inclusion for *Y*, so the only elements in both are 11, 23, 35, and 47.

2. Since any perfect square is congruent to 0 or 1(mod 4) (depending on if the square is even or odd, respectively), and since $33 \equiv 1 (\text{mod} 4)$, $x^2$ must be odd, meaning $x^2 - 33$ must be even (and thus so must be its square root); therefore, $x^2 - 33 = 4k^2 \Rightarrow 33 = x^2 - 4k^2 = (x-2k)(x+2k)$ where the factor on the left is the lesser factor. The only possible factorings of 33 are $1 \cdot 33$ and $3 \cdot 11$, making the values of $x$ 17 and 7, respectively. Thus, there are two values that work.

3. $A \cup B = B$ and $A \cap B = A$ both imply $A \subseteq B$. In fact, choice D is the only statement that is actually an accurate statement, independent of the problem.

4. $2^A$ is the power set of *A*, so its elements are subsets of *A*. Choice A is not a subset of *A*, it is an element of *A*.

5. Statement I is false (for example, $\omega_0 + 1$ is an ordinal that is not a cardinal). Statement II is true (each size of a set has a corresponding ordinal).

6. Since *X* contains 4 elements, it has a total of $2^4 = 16$ subsets. However, that includes *X* as a subset of *X*, which is not a proper subset. Therefore, there are 15 proper subsets of *X*.

7. Each set has cardinality $\omega_0$ as they are all countably infinite. Therefore, equal signs should be placed in each blank.

8. Assume *Z* is a set. According to the generalized Cantor diagonal argument, $|Z| < |2^Z|$. However, $2^Z$ is a subset of *Z* since every set in $2^Z$ is also in *Z*—this implies that $|2^Z| \leq |Z|$, and these two inequalities together give a contradiction. Therefore, *Z* is not a set.

9. Suppose that $X \neq \varnothing$ and $X \subseteq 2^X$. By the axiom of regularity, there is an element $a \in X$ such that $a \cap X = \varnothing$. However, since $a \in X \subseteq 2^X$, $a \in 2^X$, implying that $a \subseteq X$. The only way that $a \cap X = \varnothing$ and $a \subseteq X$ is if $a = \varnothing$. So the condition that must hold is that $\varnothing \in X$. Choice A is only true for $X = \varnothing$, a contradiction. Choice B is never true. The above proof shows that

$X = \{\varnothing\}$ would suffice, so choice C is not necessarily true, and therefore choice D is also not necessarily true.

10. *T* contains all the things that are not jelly doughnuts, so because jelly doughnuts exist, $T \neq \varnothing$ is correct. *T* itself is not a jelly doughnut, so $T \in T$ is not true while $T \notin T$ is true. Further, choice C is always correct, so choice B is the incorrect statement.

11. If an ordinal $\alpha \neq \varnothing$, then it is either a successor ordinal (meaning there is an ordinal immediately preceding it) or a limit ordinal (meaning there is no ordinal immediately preceding it). Therefore, $\omega_0$ is a limit ordinal.

12. $\omega_0 + 1$ is all the finite ordinals followed by $\omega_0$, so there is no way to write that other than as $\omega_0 + 1$. However, $\omega_0 + 1$ contains only one more element than $\omega_0$, whose cardinality is $\omega_0$, so the cardinality of $\omega_0 + 1$ is $\omega_0$. Choice C is the correct answer.

13. $1 + \omega_0$ is the empty set followed by all of the finite ordinals, so this is essentially $\omega_0$ (this is how ordinal addition works when at least one of the ordinals is infinite). Additionally, $1 + \omega_0$ contains only one more element than $\omega_0$, whose cardinality is $\omega_0$, so the cardinality of $1 + \omega_0$ is $\omega_0$. Choice A is the correct answer.

14. For cardinal addition, which is slightly different than ordinal addition but is the same in this instance as the previous question, the sum of the two cardinals must be the greater of the two. Choice C is the correct answer.

15. By the continuum hypothesis, every set between $\omega_0$ and $2^{\omega_0}$ must have cardinality equal to $\omega_0$, which is countably infinite. Therefore, $\omega_1 = 2^{\omega_0}$ must be uncountably infinite (choice A is incorrect because although $\omega_1$ is infinite, it is not the least cardinal that is).

16. Because numbers in the Cantor Set come from either the first third or final third of any given interval found at any iteration of the process of coming up with the Cantor Set, any number in the Cantor Set can be written in the form $\sum_{n=1}^{\infty} \frac{c_n}{3^n}$, where $c_n = 0 \text{ or } 2$ for all positive integers *n*. $\frac{1}{4} = \sum_{n=1}^{\infty} \frac{2}{3^{2n}}$ (which is of this form), $\frac{3}{10} = \sum_{n=1}^{\infty} \left( \frac{2}{3^{4n-2}} + \frac{2}{3^{4n-1}} \right)$ (which is of this form), $\frac{10}{13} = \sum_{n=1}^{\infty} \left( \frac{2}{3^{3n-2}} + \frac{2}{3^{3n}} \right)$ (which is of this form), and $\frac{27}{40} = \sum_{n=1}^{\infty} \frac{2}{3^{3n-2}}$ (which is of this form).

Therefore, all of the given answer choices are in the Cantor Set. Surprisingly, none of these numbers are endpoints of an interval left over from the removal process…even more surprisingly, the Cantor Set has an uncountable number of such elements while only having a countable number of interval endpoints. Peculiar indeed!

17. $A \subseteq A \cup B = A \cap B \subseteq B \subseteq A \cup B = A \cap B \subseteq A \Rightarrow$ all of these sets are equal; thus, $A = B$.

18. A group does not need to be commutative (which is choice D); however, if it is, it is called an Abelian group.

19. It is D—that is a lot of elements! http://mathworld.wolfram.com/MonsterGroup.html

20. $A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$, so it is all the elements that are in exactly one of *A* and *B*. Considering the given sets, the elements that satisfy this are 4, 5, 6, 7, 8, and 9, so the answer is choice C.

21. $(A \,\Delta\, (B \setminus C)) = \{2,3,5,7,8\} \,\Delta\, \{2,6\} = \{3,5,6,7,8\}$. $(A \,\Delta\, (B \,\Delta\, C)) = \{2,3,5,7,8\} \,\Delta\, (\{2,3,4,6,9\} \,\Delta\, \{3,4,5,7,9\}) = \{2,3,5,7,8\} \,\Delta\, \{2,5,6,7\} = \{3,6,8\}$. Therefore, $\{3,5,6,7,8\} \,\Delta\, \{3,6,8\} = \{5,7\}$.

22. $\left( \bigcup_{n \in \{A,B,C\}} n \right) \Delta \left( \bigcap_{n \in \{A,B,C\}} n \right) = \{2,3,4,5,6,7,8,9\} \,\Delta\, \{3\} = \{2,4,5,6,7,8,9\}$, which is equivalent to choice C.

23. $n+1 = \{0,1,2,\ldots.n-1,n\} = \{0,1,2,\ldots,n-1\} \cup \{n\} = n \cup \{n\}$, which shows that choice D is true, but it also shows that both choices A and C are also true. To see why choice B is not true, $n \,\Delta\, (n+1) = \{0,1,2\ldots,n-1\} \,\Delta\, \{0,1,2,\ldots,n-1,n\} = \{n\} \neq n$.

24. Symbolically, these are the statements $(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$, which are known as De Morgan's Laws.

25. Since baseball is solely mental and physical, all 100% of baseball is made up of these two aspects. However, the individual aspects add to 90% + 50% = 140%, so there must be exactly a 40% overage, meaning 40% must be both (if it was less than 40% for both, the percentages would sum to greater than 100%; if it was more, they would sum of less than 100%). This is really just a simple application of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$: 100% = 90% + 50% $-X$.

26. This is actually a fairly complicated problem to wrap your head around, especially if you try drawing a Venn diagram! However, we can look at it in the following way, referring to the games by their first letters:

$n(E\cup M\cup B\cup C)=n(E)+n(M)+n(B)+n(C)-n(E\cap M)-n(E\cap B)-n(E\cap C)-n(M\cap B)$
$-n(M\cap C)-n(B\cap C)+n(E\cap M\cap B)+n(E\cap M\cap C)+n(E\cap B\cap C)+n(M\cap B\cap C)$
$-n(E\cap M\cap B\cap C)\Rightarrow 66=30+34+31+26-X+5+3+5+5-1\Rightarrow X=72$, but this is the number of people who play at least two games, not exactly two games. In this counting, we counted the two-game-only intersections once each, the three-game-only intersections three times each, and the four-game-intersection six times (draw a Venn diagram to convince yourself this is the case). Each three-game-only intersection is the three-game-intersection from the table minus the four-game intersection, so the number of three-game-only intersections is $(5-1)+(5-1)+(3-1)+(5-1)=14$. Therefore, $72=Y+3(14)+6(1)\Rightarrow Y=24$, which is the number we are looking for.

27. Clearly, all choices satisfy $\varnothing\in T$ and $\mathrm{X}\in T$. To check unions and intersections, since *X* has a finite number of elements, you only need check that the union or intersection of any two element of *T* is itself an element of *T*, and you don't need to check either $\varnothing$ or X in those unions or intersections. Choice B is clearly not a topology on *X* since it does not contain the union of the two singletons. To see that the other choices are topologies,

Choice A: $\{a,b\}\cup\{c\}=X$; $\{a,b\}\cap\{c\}=\varnothing$

Choice C: $\{c\}\cup\{a,c\}=\{a,c\}$, $\{c\}\cup\{b,c\}=\{b,c\}$, $\{a,c\}\cup\{b,c\}=X$; $\{c\}\cap\{a,c\}=\{c\}\cap\{b,c\}$
$=\{a,c\}\cap\{b,c\}=\{c\}$

Choice D: $\{a\}\cup\{c\}=\{a\}\cup\{a,c\}=\{c\}\cup\{a,c\}=\{a,c\}$, $\{a\}\cup\{b,c\}=\{a,c\}\cup\{b,c\}=X$,
$\{c\}\cup\{b,c\}=\{b,c\}$; $\{a\}\cap\{c\}=\{a\}\cap\{b,c\}=\varnothing$, $\{a\}\cap\{a,c\}=\{a\}$, $\{c\}\cap\{a,c\}=\{c\}\cap\{b,c\}$
$=\{a,c\}\cap\{b,c\}=\{c\}$

28. The power set of a set is always a topology on that set since every possible subset is an element of the topology—for *X*, there are $2^3=8$ total subsets, and this is obviously the value of *M*. The topology with the least number of elements is actually the trivial topology: $T=\{\varnothing,X\}$ (notice that this satisfies the definition for a topology)—this makes $m=2$. Therefore, $M-m=6$.

29. Multiplication of quaternions does allow for the commutation of real numbers in the product with the other basis elements. Since $ijk=-1$, $-ij=ij(-1)=ijk^2=-1(k)=-k\Rightarrow ij=k$. Therefore, $1=-j^2=j(-1)j=ji^2j=jiij=jik\Rightarrow k=1(k)=jik^2=ji(-1)=-ji\Rightarrow ji=-k$.

30. To do this efficiently, we need the other products of the basis elements, which can be found in a similar way to that of the previous problem. The other products are $ij = k$, $ik = -j$, $jk = i$, $ki = j$, and $kj = -i$. Be very careful when multiplying these two quaternions!

$(1+2i-j+k)(-2+i+j-3k) = -2-4i+2j-2k+i+2i^2-ji+ki+j+2ij-j^2+kj-3k-6ik+3jk$
$-3k^2 = -2-4i+2j-2k+i-2+k+j+j+2k+1-i-3k+6j+3i+3 = -i+10j-2k$ .