Theta Circles & Polygons 2015 Answer Key

1. C
2. E
3. D
4. B
5. B
6. C
7. A
8. A
9. D
10. D
11. C
12. C
13. D
14. A
15. B
16. D
17. A
18. A
19. A
20. B
21. B
22. C
23. A
24. C
25. C
26. A
27. C
28. A
29. B
30. B

1. Draw a radius from P to A. Now consider right ΔAPB with right angle at A. You will then have $x^2 + (2x+4)^2 = (2x+6)^2$. This produces the solution x = 10, and gives the side lengths to be 10, 24, and 26. The perimeter is 60.

2. In order to have a higher number of angles, and thus the greatest number of sides, we use as many small odd integers as possible. The sum of the first n consecutive positive odd integers is n^2. These angles must also add up to 360. Note that $19^2 = 361$. If we then take away the angle with measure 1°, then we have 360° and 18 angles, and 18 sides.

3. Area 36π gives a radius of 6. Draw radii to the end of the given chords, and you will see you have a 45-45-90 triangle. Now complete the picture by drawing an inscribed square JKHI (one side of which is the chord you just drew). This gives 4 congruent arcs. If point L is on the arc cut off by $\overline{JK}$, then ∠L is obtuse. If L is on the arc cut off by $\overline{JI}$, then ∠J is obtuse. If L is on the arc cut off by $\overline{KH}$, then ∠K is obtuse. Finally, if L is on the arc cut off by $\overline{HI}$, then the triangle is acute. Thus, the answer ¾.

4. The only regular polygons that can tessellate a plane independently are equilateral triangles, squares, and regular hexagons. The interior angles of these polygons are 60, 90, and 120 degrees, respectively, and the sum of those three is 270.

5. The radius has nothing to do with the problem. The exterior angle is 15 degrees. Since the sum of the exterior angles is 360, with one at each vertex, then there are 24 angles, and 24 vertices. This means that a diameter will touch vertices that are 12 spaces apart. N is the 14th letter of the alphabet, so subtract 12, and we get the 2nd letter – B, which thankfully is choice B as well.

6. Since the inscribed angle D has measure 50, its intercepted arc BC has measure 100. Now draw the center of the circle (say E), and the central angle BEC has measure 100. Now ABEC is a kite with right angles at B and C, and 100° angle at E. This leaves **<u>80 degrees</u>** to complete the quadrilateral.

7. Paul travels 5 spaces while Polly travels 7. The only way for them to end up back at the same point is if they travel a number of spaces that is a multiple of 12. If they both move 12 times, then Paul travels 60 spaces, while Polly travels 84. Polly has travelled 24 more spaces, which is 2 more revolutions. (It might be noted that when they both move 6 times, they are on the same vertex on the polygon, but it is not the starting vertex.)

8. Start adding up the toothpicks you need to build these polygons, and we want the biggest one we can build before we run out of toothpicks. Starting with 3, we are looking for the greatest n such that $3 + 4 + 5 + \cdots + n \leq 2015$. To make life easy, add 1 and 2 to each side. So now we have the equation, $1 + 2 + 3 + 4 + 5 + \cdots + n \leq 2015 + 1 + 2$, and this becomes

$\frac{n(n+1)}{2} \leq 2018$ or $n(n+1) \leq 4036$. The largest integer *n* that satisfies this condition is 63. An exterior angle for this polygon is $\frac{360}{63}$ or $\frac{40}{7}$.

9. Draw any two of the perpendicular bisectors. Where these intersect is the center of the circle, since any point on a perpendicular bisector of a segment is equidistant from the endpoints of the segment. The perp. bisector of the segment with endpoints (-3, -2) and (5, -2) is just the line x=1. The perp. bisector between (-3, -2) and (-9, 4) passes through the midpoint (-6, 1). The slope is perpendicular to -1, so the equation of the perp. bisector here is $y - 1 = (x + 6)$. This intersects the line x=1 when the at the point (1, 8). The distance between this point and any of the given 3 points is $\sqrt{116}$, which is the radius. No need to simplify, since all we want is the area, so $\pi r^2 = 116\pi$.

10. Let the radius of the circle be 1 unit. The circle sweeps out a sphere with volume equal to $\frac{4}{3}\pi$. The square sweeps out 2 congruent cones sharing a base, and each having a radius of 1 and a height of one, and thus a volume equal to $\frac{1}{3}\pi$ each or $\frac{2}{3}\pi$ total. The ratio is then $\frac{2}{3}\pi : \frac{4}{3}\pi$, which becomes 1:2.

11. It is a fascinating thing that a HEPTAGON has 8 letters and OCTAGON has 7 letters. 8 + 7 = 15.

12. Gauss proved that a heptadecagon (17 sides) is constructible. It is the wonderful story of modern mathematics used to solve old problems, something all of us should continue to do.

13. An exterior angle is $360 \div 10$, which is 36. Thus the interior angle is 144 (degrees, of course).

14. For two chords intersecting in a circle, the product of the two pieces on one chord is equal to the product of the two pieces on the other. So $8(18) = x(4x)$, which gives $x = 6$. Now draw perpendicular distances from E to each chord. Since CD=26, and half of that is 13, you can find the distance from E to $\overline{AB}$ is 5. Then we know that AB=30, so half of that chord is 15. Draw the radius to point A in order to complete the right triangle, and us the Pythagorean Theorem with legs 5 and 15, which gives hypotenuse = radius = $5\sqrt{10}$.

15. Jill's polygon is n-sided, which has $\frac{n(n-3)}{2}$ diagonals. Jane's has n+3 sides, and has $\frac{(n+3)n}{2}$ diagonals. Since Jane's has 3 times the diagonals of Jill's, we solve the equation: $\frac{(n+3)n}{2} = 3\left[\frac{n(n-3)}{2}\right]$, and Jill has a hexagon, and Jane has a nonagon. So 6 + 9 = 15.

16. There are (at least) two ways to do this problem. First, with law of cosines, and drawing two radii (r=1) with an included 45° angle. Then $c^2 = a^2 + b^2 - 2ab \cdot cos(C)$ becomes $side^2 = 1^2 + 1^2 - 2(1)(1) \cdot cos(45°)$ = $2 - \sqrt{2}$. When we take the square root, we get

$\sqrt{2-\sqrt{2}}$. Another way is to draw two radii, then drop an altitude from one vertex to the other radius. This makes a 45-45-90 with hypotenuse 1, and the legs would be $\frac{\sqrt{2}}{2}$. Then the smaller right triangle formed by the altitude drawn has legs $\frac{\sqrt{2}}{2}$ and $1-\frac{\sqrt{2}}{2}$. Complete the Pythagorean Theorem with these two legs, and you again arrive at a side length of $\sqrt{2-\sqrt{2}}$.

17. Draw the two externally tangent circles, as well as the common external tangent. Then draw a radius from each center perpendicular to the tangent line. Then draw two segments – one from the center of the smaller circle to the center of the larger, and one from the center of the smaller circle perpendicular to the radius of the larger. This should create a right triangle where the hypotenuse is the sum of the radii (26) and one leg is the difference of the radii (10). This is a 5-12-13 triangle multiplied by 2, and the other leg (of length 24) is the same length as the common external tangent. The answer is 24.

18. In order to round to 180 for the interior angle, the exterior angle must be less than or equal to 0.5. So we solve the equation $\frac{360}{n} \leq 0.5$, which gives $n \geq 720$. So 7+2+0 = 9.

19. One way to think of it is to draw the circumscribed circle. Consider that each point of the star is an inscribed angle, and all 5 of the intercepted arcs form the entire circle. Since each inscribed angle is half of the intercepted arc, then the sum of the 5 is half of 360, which is 180.

20. If all 4 triangles have a common vertex, you can draw around a little, but you will find that this is only possible if each of the angles at the common vertex is a right angle. Since the legs are not required to be the same length, they do not have to be isosceles right triangles, so right triangles are the best we can do. This forms a rhombus.

21. I can form a triangle with any group of 3 points that is not collinear. The number of groups of 3 formed from these 10 points is ${}_{10}C_3$ or 120. However, the 4 points on the diameter cannot produce a triangle, so we remove these groups of 3 points. So we remove ${}_4C_3$ or 4, and we get 116 possible triangles.

22. Draw the circumscribed circle. Notice that ∠AFD intercepts 3/8 of the circle, which is 135 degrees. Since this angle is inscribed, its measure is 135/2 or 67.5.

23. The number of diagonals drawn from each vertex is 3 less than the number of sides, so the number of sides here is d+3. Draw d diagonals from each of the d+3 vertices, then divide by 2 since this process would draw every diagonal in both directions. So the total number of diagonals is $\frac{(d+3)d}{2}$, which is choice A.

24. Draw the diagonals of the rhombus, which form right triangles. Since the perimeter of the rhombus is 100, each side, and each hypotenuse is 25. The diagonal of length 14 is bisected to

give the right triangles one leg of length 7. The other leg is 24, which makes the other diagonal 48. The area of the rhombus is half the product of the diagonals, which is $\frac{1}{2}(14)(48)$ or 336.

25. Let the radius of one of the circles be *r*. Then the area of the 4 circles combined is $4\pi r^2$. The side length of the square is 4r, giving the area of the square to be $16r^2$. Then the area outside the circles but inside the square is $16r^2 - 4\pi r^2$. So the probability of landing in this area is $\frac{16r^2-4\pi r^2}{16r^2}$ or $\frac{4-\pi}{4}$. Notice that this is the same as if you had just considered one circle and its circumbscribed square.

26. Opposite angles of an inscribed quadrilateral are supplementary, so the pair of them cuts off the entire 360° together. This means that the 250 and 110 are intercepted arcs for opposite angles, and the arc intercepted by the angle opposite the angle intercepting 130 must be a 230 degree arc.

27. Draw the radius AG. We know AE=4 and EG=3, so AG=5. If we use this radius for the other chord, and GF=4, then DF=3, and CD=6.

28. The perimeter is $\frac{1}{4}$ of the original octagon, which means the area is $\left(\frac{1}{4}\right)^2$ or $\frac{1}{16}$ of the original. One-sixteenth of 432 is 27.

29. We know the sum of the interior angles of a polygon will have to be a multiple of 180. The smallest multiple of 180 which is greater than 1524 is 1620. This means that the remaining angle must be
1620 – 1524 = 96. Then 9 x 6 = 54.

30. So, in order to draw a star, draw diagonals to vertices in a clockwise fashion, traversing the same number of "steps" (say *k*) each time. When doing this, you should have 2 discoveries: (1)if k=1 this is not a diagonal at all, but a side for the n-gon, and (2)that if *k* is relatively prime to *n*, then you will not return to your original vertex until you have hit every other vertex exactly once. Then you are looking for k-values that are relatively prime to n. Finally, you discover that k must be less than one-half of n in order to prevent duplication (you end up with a complementing scenario – try letting k=2 for a 5-pointed star, and you can see that k=3 gives a congruent star). So we are looking for k-values relatively prime to 13 and less than one-half of 13. Acceptable k-values are 2, 3, 4, 5, and 6. There are 5 values, and 5 ways to draw a 13-pointed star.